Appendix to the Articles

Construction of Blow-up Sequences for the Prescribed Scalar Curvature Equation on S^n. I and II.

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In this appendix we follow the notations, conventions, equation numbers, section numbers, lemma, proposition and theorem numbers as used in the main articles (Part I [25] and Part II [26]), *unless otherwise is specifically mentioned* (*for instances, those equation numbers starting with* A). *The references for the appendix are listed at the last three pages.*

§ A. 1 A proof of the integration by parts formula (2.11) in Part I.

The Hilbert space $\mathcal{D}^{1,2}$ coincides with the completion of $C_o^\infty(\mathbb{R}^n)$ with respect to the L^2-norm of the gradient [1]. That is, for any $f \in \mathcal{D}^{1,2}$, there exists a sequence of compactly supported smooth functions $\{f_{o,i}\}$ so that

$$(A.1.1) \qquad \| f - f_{o,i}\|_{\bigtriangledown}^2 = \int_{\mathbb{R}^n} |\bigtriangledown (f - f_{o,i})|^2 \to 0 \quad \text{as } i \to \infty\,.$$

Using the Green identity (compact support $\Longrightarrow$ no boundary term), we find

$$\int_{\mathbb{R}^n} \langle \bigtriangledown f_{o,i}\,,\, \bigtriangledown h\rangle = -\int_{\mathbb{R}^n} f_{o,i}\cdot \Delta_o h \qquad \text{for} \quad i = 1,\ 2, \cdots.$$

It follows that

$$\left| \int_{\mathbb{R}^n} \langle \nabla f \,,\, \nabla h \rangle \;+\; \int_{\mathbb{R}^n} (f\,\Delta_o\,h) \right|$$

$$\leq \left| \int_{\mathbb{R}^n} \{ \langle \nabla (f - f_{o,i}) \,,\, \nabla h \rangle + \langle \nabla f_{o,i} \,,\, \nabla h \rangle + (f_{o,i}\,\Delta\,h) + ([\,f - f_{o,i}\,]\,\Delta_o\,h) \} \right|$$

$$\leq \int_{\mathbb{R}^n} |\,\nabla\, (f - f_{o,i})| \cdot |\,\nabla\, h| \;+\; \int_{\mathbb{R}^n} |\,f - f_{o,i}| \cdot |\,\Delta_o\,h\,|$$

$$\leq \left[\int_{\mathbb{R}^n} |\,\nabla\, (f \,-\, f_{o,i})|^2 \right]^{\frac{1}{2}} \cdot \left[\int_{\mathbb{R}^n} |\,\nabla\, h|^2 \right]^{\frac{1}{2}} \qquad (\,\to 0 \quad \text{as} \quad i \,\to\, \infty)$$

$$+ \left[\int_{\mathbb{R}^n} |\,f \,-\, f_{o,i}|^{\frac{2n}{n-2}} \right]^{\frac{n-2}{2n}} \cdot \left[\int_{\mathbb{R}^n} |\,\Delta_o\,h|^{\frac{2n}{n+2}} \right]^{\frac{n+2}{2n}} \qquad (\,\to 0 \quad \text{as} \quad i \,\to\, \infty)\,.$$

Here we use (A.1.1), Hölder's inequality, the Sobolev inequality (2.9) , and (2.10), both in Part I . □

§ A. 2. Riesz representation theorem.

For every bounded linear functional $F : \mathcal{H} \to \mathbb{R}$ on a Hilbert space $\mathcal{H}$ over the scalar field $\mathbb{R}$, equipped with inner product $\langle \;,\; \rangle_{\mathcal{H}}$, there is a unique element $v \in \mathcal{H}$ so that

$$(A.2.1) \qquad F\,(u) \;=\; \langle u \,,\; v \rangle_{\mathcal{H}} \qquad \text{for all} \quad u \,\in\, \mathcal{H}\,.$$

Moreover, we may take

$$(A.2.2) \qquad v \;=\; \frac{F(w)}{\|\,w\|_{\mathcal{H}}^2} \cdot w \qquad \text{for all} \quad w \,\perp\, \mathbf{N} \,:=\, \{\,F(\bullet) \,=\, 0\}\,,$$

and $\|\,F\|_{\mathcal{H}}^2 = \langle v \,,\; v\,\rangle_{\mathcal{H}}$. See, for example, [27] .

Corollary A.2.3. *Under the conditions and notations described in* (A.2.1) *and* (A.2.2), *suppose* $w \,\in\, \mathcal{H}$ *with* $F(w) \,\neq\, 0$, *then*

$$(A.2.4) \qquad \|\,F\|^2 \;\geq\; \frac{|\,F(w)|^2}{\|\,w\|_{\mathcal{H}}^2}\,.$$

(*Here* w *may not be perpendicular to the null space of* F *.*)

Proof. We have

$$w \;=\; w_\perp + w_o \quad \text{with} \;\; w_\perp \perp \mathbf{N} \;\; \text{and} \;\; w_o \in \mathbf{N} \;\; \Longrightarrow \;\; \|\, w\|^2_{\mathcal{H}} \;=\; \|\, w_\perp\|^2_{\mathcal{H}} + \|\, w_o\|^2_{\mathcal{H}}.$$

Hence

$$\|F\|^2 \;=\; \frac{|\, F(w_\perp)|^2}{\|\, w_\perp\|^2_{\mathcal{H}}} \;=\; \frac{|\, F(w)|^2}{\|\, w_\perp\|^2_{\mathcal{H}}} \;\geq\; \frac{|\, F(w)|^2}{\|\, w\|^2_{\mathcal{H}}}.$$

The result follows. □

Regarding the functional I_o in (2.13), Part I, consider the following. For $\mathbf{z} \in \mathbf{Z}$,

$$(A.2.5) \qquad \begin{aligned} v \in \operatorname{Ker} I_o''(\mathbf{z}) \quad &\Longrightarrow \quad (I_o''(\mathbf{z})\,[v]\,f) \;=\; 0 \qquad \text{for all} \;\; f \in \mathcal{D}^{1,2} \\ &\Longrightarrow \quad (I_o''(\mathbf{z})\,[f]\,v) \;=\; 0 \qquad [\,\text{using} \;\; (2.21)\,, \;\; \text{Part} \;\; \text{I}\,] \\ &\Longrightarrow \quad v \in \operatorname{Ker} I_o''(\mathbf{z})\,[f] \\ &\Longrightarrow \quad \langle h,\; v\rangle_{\triangledown} \;=\; 0 \\ &\qquad\qquad [\,\text{for} \;\; h\,, \;\; \text{refer} \;\; \text{to} \;\; (2.23)\,, \;\; \text{Part} \;\; \text{I}\,] \\ &\Longrightarrow \quad h \;\perp\; \operatorname{Ker} I_o''(\mathbf{z})\,. \end{aligned}$$

Moreover,

$$(A.2.6) \qquad \begin{aligned} u \in \operatorname{Coker} I_o''(\mathbf{z}) \quad &\Longleftrightarrow \quad \left\langle \frac{(I_o''(\mathbf{z})\,[f]\,h)}{\|h\|^2_{\triangledown}} \cdot h\,,\; u \right\rangle_{\triangledown} \;=\; 0 \\ &\Longleftrightarrow \quad (I_o''(\mathbf{z})\,[f]\,u) \;=\; 0 \qquad \text{for all} \;\; f \;\in\; \mathcal{D}^{1,2} \\ &\Longleftrightarrow \quad (I_o''(\mathbf{z})\,[u]\,f) \;=\; 0 \qquad \text{for all} \;\; f \;\in\; \mathcal{D}^{1,2} \\ &\qquad\qquad [\,\text{via} \;\; (2.21)\,, \;\; \text{Part} \;\; \text{I}\,] \\ &\Longleftrightarrow \quad u \;\in\; \operatorname{Ker} I_o''(\mathbf{z})\,. \end{aligned}$$

This symmetry allows us to show that $I_o''(u)$, treated in the above way, is a Fredholm map with index zero once we know that the kernel has finite dimension. See Lemma 4.1 in [3], pp. 47 − 50.

§ A. 3 Non-degenerate critical points and fundamental notions in the degree theory.

§ A.3. a. *Jacobian and the degree of a regular value for a C^1 - map.* See [14], pp. 6. Here we restrict ourselves to an open and bounded domain Ω in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Let

$$\begin{array}{ccc} \Phi : \overline{\Omega} & \rightarrow & \mathbb{R}^n \\ \cap & & \\ \mathbb{R}^n & & \end{array}$$

be a C^1-map (up to the boundary $\partial\Omega$). Suppose $\vec{0} \in \Phi(\Omega) \setminus \Phi(\partial\Omega)$ is a regular value (pre-images having non-zero Jacobians), then

$$\text{Deg}\,(\Phi\,,\ \Omega\,,\ \vec{0}) \ := \sum_{y\, \in\, \Phi^{-1}(\vec{0})} \text{sign}\, J_\Phi(y)\,. \tag{A.3.1}$$

Here J_Φ is the Jacobian (determinant) of the map Φ.

Intuitively, the degree counts the 'left/right' cutting of the map at $\vec{0}$. In case $\vec{0}$ is a not a regular value, the degree is defined by using a "near-by" regular value. See [14].

§ A.3. b. *Degree for a continuous function.* Let

$$\Phi_o : \overline{\Omega} \rightarrow \mathbb{R}^n \tag{A.3.2}$$

be C^o (up to the boundary) and

$(A.3.3)$ $\quad \vec{0} \notin \Phi_o(\partial\Omega)$. Indeed, assume that $\delta := \text{dist}\,(\vec{0}\,,\ \Phi_o(\partial\Omega)) > 0$.

Define

$$\text{Deg}\,(\Phi_o\,,\ \Omega\,,\ \vec{0}) \ = \ \text{Deg}\,(\Psi\,,\ \Omega\,,\ \vec{0})$$

for any $\Psi \in C^1(\overline{\Omega})$ such that

$$|\,\Phi_o(y) - \Psi(y)| \ < \ \delta \quad \text{for all } \ y \in \overline{\Omega} \ = \ \Omega \cup \partial\Omega\,.$$

See Definition 1.18 in [14], pp. 17.

§ A.3. c. *C^o property of degree and existence of solution.* Let Φ_o be as in (A.3.2) satisfying (A.3.3). Then we have the following properties. (See Theorems 2.1 and 2.3 in [14], pp. 30.)

$$\text{Deg}\,(\,\Phi_o\,,\ \Omega\,,\ \vec{0}) \ \neq \ 0 \ \Longrightarrow \ \Phi_o(y_o) \ = \ \vec{0} \quad \text{for a } \ y_o \in \Omega\,. \tag{A.3.4}$$

That is, the equation $\Phi_o(y) = \vec{0}$ has a solution $y_o \in \Omega$.

$$(A.3.5) \qquad \text{For any } \Psi_o \in C^o(\overline{\Omega}) \text{ with } \| \Phi_o - \Psi_o \|_{C^o(\Omega \cup \partial\Omega)} < \delta$$

$$\Longrightarrow \qquad \mathrm{Deg}(\Phi_o, \Omega, \vec{0}) = \mathrm{Deg}(\Psi_o, \Omega, \vec{0}).$$

Here δ is given in (A.3.3).

§ A.3. d. *Hessian and stable critical point of a C^2-function.* Let $\mathcal{O}$ be a non-empty open set in $\mathbb{R}^n$ and

$$f : \mathcal{O} \to \mathbb{R}$$

be a C^2-function. Suppose $y_c \in \mathcal{O}$ a critical point for f (i.e., $\bigtriangledown f(y_c) = \vec{0}$). y_c is called a *stable critical point* if there is an open bounded domain $\Omega \subset\subset \mathcal{O}$ such that $y_c \in \Omega$ and

$$\mathrm{Deg}(\bigtriangledown f, \Omega, \vec{0}) \neq 0.$$

(We assume that $\partial\Omega$ is smooth.) Here we render

$$\bigtriangledown f : \mathcal{O} \subset \mathbb{R}^n \to \mathbb{R}^n$$

$$y \mapsto \bigtriangledown f(y).$$

Refer to [3], pp. 25. The Jacobian of the map $\bigtriangledown f$ is then the determinant of the Hessian matrix of the function f. See, for example, [32], pp. 93.

A critical point of a C^2-function f is called non-degenerate if the Hessian is a *non-degenerate* bilinear form, or equivalently, the determinant of the Hessian is non-zero. As a consequence, the critical point is isolated. That is, there is a ball $B_{y_c}(r)$ such that there is no other critical point in $B_{y_c}(r)$ besides y_c. Take $\Phi = \bigtriangledown f$ in (A.3.1), we obtain

(A.3.6)

$$\mathrm{Deg}(\bigtriangledown f, B_{y_c}(r), \vec{0}) = \mathrm{sign}\, J_{\bigtriangledown f}(y_c) = \frac{\text{det Hessian at } y_c}{|\text{det Hessian at } y_c|} \neq 0 \Longrightarrow \text{“stable”}.$$

Moreover, let

$$(A.3.7) \qquad \bar{\delta} := \min_{y \in \partial B_{y_c}(r)} |\bigtriangledown f(y)|.$$

For a C^1-function h defined in a neighborhood of $\overline{B_{y_c}(r)}$, via (A.3.5), we have

$$(A.3.8) \qquad \text{“} |\bigtriangledown f(y) - \bigtriangledown h(y)| < \bar{\delta} \quad \text{for all } y \in \overline{B_{y_c}(r)} \text{”}$$

$$\Longrightarrow \qquad \text{“}\mathrm{Deg}(\bigtriangledown f, B_{y_c}(r), \vec{0}) = \mathrm{Deg}(\bigtriangledown g, B_{y_c}(r), \vec{0})\text{”}.$$

§ A. 4. Curvature, gradient, Hessian, cancelation & matching.

§ A.4.a. *Calculation of the sectional curvature.* Refer to § 3 b in Part I. On $\mathbb{R}^+ \times \mathbb{R}^n$ with coordinates

$$(\lambda\,,\ \xi_1\,, \cdots,\ \xi_n)\,,$$

indexed zero $\uparrow$

we denote

$$\partial_o \ = \ \frac{\partial}{\partial \lambda}\,, \quad \partial_1 \ = \ \frac{\partial}{\partial \xi_1}\,, \cdots, \quad \partial_n \ = \ \frac{\partial}{\partial \xi_n}\,.$$

Consider a metric of the form

$$g_{\mathbf{z}_{ij}} \ = \ \frac{c_i\,\delta_{ij}}{\lambda^2}\,, \quad 0 \le i\,,\ j \le n\,, \ \text{ with } \ c_o \ = \ \bar{c}_o > 0 \ \text{ and } \ c_1 = c_2 = \cdots = c_n = \bar{c}_1 > 0\,.$$

In [3] , pp. 73 - 74, we find a list of standard formulas for Riemannian curvature tensor. Let us start with the Christoffel symbol, which is given by

$$\Gamma^w_{ij} \ = \ \frac{1}{2}\,[\,D_i\,h_{wj} + D_j\,h_{wi} - D_w\,h_{ij}\,] \cdot g_{\mathbf{z}}{}^{ww}$$

$$\Longrightarrow \ \Gamma^o_{jj} \ = \ \frac{c_o^{-1}\,c}{\lambda}\,, \qquad \Gamma^o_{oo} = \Gamma^w_{ow} = -\frac{1}{\lambda}\,, \ \ \Gamma^w_{oo} = \Gamma^o_{io} = \text{others} = 0 \qquad \text{for } \ w \ne 0, \quad i \ne 0\,.$$

Recall that

$$R\,(\partial_u,\ \partial_v)\,\partial_w = \nabla_{\partial_u}\,\nabla_{\partial_v}\,\partial_w - \nabla_{\partial_v}\,\nabla_{\partial_u}\,\partial_w - \nabla_{[\partial_u\,,\ \partial_v]}\,\partial_w\,.$$

In local coordinates x^μ the Riemann curvature tensor is given by

$$R^w{}_{jkl} \ = \ dy^w(R(\partial_k,\ \partial_l)\,\partial_j)\,,$$

$$R^w{}_{jkl} \ = \ D_k\,\Gamma^w_{lj} - D_l\,\Gamma^w_{kj} + \Gamma^w_{km}\,\Gamma^m_{lj} - \Gamma^w_{lm}\,\Gamma^m_{kj}$$

$$\Longrightarrow \ R^w{}_{jij} \ = \ D_i\,\Gamma^w_{jj} - D_j\,\Gamma^w_{ij} + \Gamma^w_{im}\,\Gamma^m_{jj} - \Gamma^w_{jm}\,\Gamma^m_{ij} = \Gamma^w_{io}\,\Gamma^o_{jj} - \Gamma^w_{jo}\,\Gamma^o_{ij}$$

$$= \ \left(\frac{c_o^{-1}\,c}{\lambda}\right)\left(-\frac{1}{\lambda}\right) \qquad\qquad (\,i \ne 0\,, \quad j \ne 0\,, \quad i \ne j\,, \quad w \,=\, i)\,.$$

The related covariant tensor is obtained via

$$R_{ijkl} \ = \ g_{\mathbf{z}iw}\,R^w{}_{jkl}$$

$$\Longrightarrow \ R_{ijij} \ = \ g_{\mathbf{z}iw}\,R^w{}_{jij} \ = \ \frac{c}{\lambda^2}\left(\frac{c_o^{-1}\,c}{\lambda}\right)\left(-\frac{1}{\lambda}\right)$$

$$(\,i \ne 0\,, \quad j \ne 0\,, \quad i \ne j\,)\,.$$

The sectional curvature [21] is given by

$$
\begin{aligned}
K(i,\,j) \;&=\; \frac{\langle\, R(\partial_i\,,\ \partial_j)\,\partial_j\,,\ \partial_i\rangle_{g_{\mathbf{z}}}}{\langle\,\partial_i\,,\ \partial_i\rangle_{g_{\mathbf{z}}}\langle\,\partial_j\,,\ \partial_j\rangle_{g_{\mathbf{z}}} - \langle\,\partial_i\,,\ \partial_j\rangle^2_{g_{\mathbf{z}}}} = \frac{g_{\mathbf{z}iw}\,R^w{}_{jij}}{g_{\mathbf{z}ii}\,g_{\mathbf{z}jj} - g_{\mathbf{z}}{}^2_{ij}} \\
\Longrightarrow\quad K(i,\,j) \;&=\; \frac{R_{ijij}}{g_{\mathbf{z}ii}\,g_{\mathbf{z}jj} - g_{\mathbf{z}}{}^2_{ij}} = -\,\frac{1}{c_o} \qquad \text{for}\quad i \neq 0\,,\quad j \neq 0\,,\quad i \neq j\,.
\end{aligned}
$$

$$
\begin{aligned}
R_{ojoj} \;&=\; g_{\mathbf{z}oo}\,R^o{}_{joj}\,, \\
R^o{}_{joj} \;&=\; \frac{\partial\,\Gamma^o_{jj}}{\partial\lambda} - 0 \,+\, \Gamma^o_{oo}\,\Gamma^o_{jj} - \Gamma^o_{jj}\,\Gamma^j_{oj} \\
&=\; -\left(\frac{c_o^{-1}\,c}{\lambda^2}\right) - \left(\frac{c_o^{-1}\,c}{\lambda^2}\right) + \left(\frac{c_o^{-1}\,c}{\lambda^2}\right) = -\left(\frac{c_o^{-1}\,c}{\lambda^2}\right) \\
\Longrightarrow R_{ojoj} \;&=\; -\,\frac{c_o}{\lambda^2}\left(\frac{c_o^{-1}\,c}{\lambda^2}\right) \\
K(0,\,j) \;&=\; -\,\frac{1}{c_o} \qquad\quad \text{for}\quad j \,\neq\, 0\,.
\end{aligned}
$$

§ A.4.b. *Gradient and Hessian.* It follows from the standard expression for gradient [33] [21] (pp. 28) that

$$
\bigtriangledown_{g_{\mathbf{z}}}\mathcal{F} \;=\; g_1^{ij}\;[\,\partial_i\,\mathcal{F}]\;\partial_j \;=\; \lambda^2\left(\frac{1}{c_o}\frac{\partial\,\mathcal{F}}{\partial\lambda}\,\partial_o \;+\; \frac{1}{c_1}\frac{\partial\,\mathcal{F}}{\partial\xi_1}\,\partial_1 \;+\cdots+\; \frac{1}{c_1}\frac{\partial\,\mathcal{F}}{\partial\xi_n}\,\partial_n\right)
$$

$$
\Longrightarrow \qquad \bigtriangledown_{g_{\mathbf{z}}}\mathcal{F}(\bar{p}) \;=\; 0 \quad\Longleftrightarrow\quad \frac{\partial\,\mathcal{F}}{\partial\lambda}(\bar{\lambda}\,,\ \bar{\xi}) = \frac{\partial\,\mathcal{F}}{\partial\xi_1}(\bar{\lambda}\,,\ \bar{\xi}) = \cdots = 0\,.
$$

Here,

$(A.4.1)$ the image of $(\bar{\lambda}\,,\ \bar{\xi})$ is denoted by $\bar{p}$ via the parametrization.

The Hessian is given by [21]

$$
\mathrm{Hess}_{\,g_{\mathbf{z}}}\,\mathcal{F} \;=\; \bigtriangledown^2_{g_1}\mathcal{F} \;=\; (\bigtriangledown_{g_{\mathbf{z}}}\mathcal{F})^i_{\,;\,j}\;\partial_i\otimes dy^j
$$

with
$$
(\bigtriangledown_{g_{\mathbf{z}}}\mathcal{F})^i_{\,;\,j} \;=\; \partial_j\,(\bigtriangledown_{g_{\mathbf{z}}}\mathcal{F})^i \;+\; (\bigtriangledown_{g_{\mathbf{z}}}\mathcal{F})^k\,\Gamma^i_{jk}\,,
$$

where Γ^i_{jk} is the Christoffel symbol [21]. In particular,

$$
\begin{aligned}
\nabla_{g_{\mathbf{z}}} \mathcal{F}(\bar p) \ &= \ 0 \\
\Longrightarrow \ (\nabla_{g_{\mathbf{z}}} \mathcal{F})^i_{\ ;j}\,(\bar p) \ &= \ \left[\frac{\partial^2 \mathcal{F}\,(\bar\lambda\,,\ \bar\xi)}{\partial \xi_i\ \partial \xi_j} \right] \partial_i \otimes dy^j \qquad \text{for} \ \ i \neq 0 \ \ \text{and} \ \ j \neq 0\,, \\
(\nabla_{g_{\mathbf{z}}} \mathcal{F})^i_{\ ;j}\,(\bar p) \ &= \ \left[\frac{\partial^2 \mathcal{F}\,(\bar\lambda\,,\ \bar\xi)}{\partial \xi_j\ \partial \lambda} \right] \partial_o \otimes dy^j \qquad \text{for} \ \ i \,=\, 0\,, \ \ j \,\ge\, 1\,, \\
(\nabla_{g_{\mathbf{z}}} \mathcal{F})^0_{\ ;0}\,(\bar p) \ &= \ \left[\frac{\partial^2 \mathcal{F}\,(\bar\lambda\,,\ \bar\xi)}{\partial \lambda^2} \right] \partial_o \otimes d\lambda \qquad \text{for} \ \ i \,=\, 0\,, \ \ j \,\ge\, 1\,,
\end{aligned}
$$

$$
\Longrightarrow \ \text{Hess}_{\,g_{\mathbf{z}}}\, \mathcal{F}(\bar p) \ \ \text{is} \ \ \ \text{non-degenerate} \ \ \Longleftrightarrow \ \ \text{Hess}_{\,(\lambda,\ \xi)}\ \mathcal{F}\ (\bar\lambda\,,\ \bar\xi) \ \text{is} \ \ \text{non-degenerate}\,.
$$

§ A.4. c. Refer to § 3 d, Part I. First we find

$$
\Delta_y \left[\frac{\lambda^2 - r^2}{(\lambda^2 + r^2)^{\frac{n}{2}}} \right] \qquad\qquad (\, r^2 = y_1^2 + \cdots + y_n^2\,)\,.
$$

$$
\frac{\partial}{\partial r} \left[\frac{\lambda^2 - r^2}{(\lambda^2 + r^2)^{\frac{n}{2}}} \right] \ = \ \frac{-2r}{(\lambda^2 + r^2)^{\frac{n}{2}}} - n \frac{r\,(\lambda^2 - r^2)}{(\lambda^2 + r^2)^{\frac{n}{2}+1}}
$$

$$
\frac{\partial^2}{\partial r^2} \left[\frac{\lambda^2 - r^2}{(\lambda^2 + r^2)^{\frac{n}{2}}} \right] \ = \ \frac{-2}{(\lambda^2 + r^2)^{\frac{n}{2}}} - \frac{n\,\lambda^2}{(\lambda^2 + r^2)^{\frac{n}{2}+1}} + \frac{3n\,r^2}{(\lambda^2 + r^2)^{\frac{n}{2}+1}}
$$

$$
+ \ \frac{2n\,r^2}{(\lambda^2 + r^2)^{\frac{n}{2}+1}} \ + \ \frac{n\,(n+2)\,r^2\,(\lambda^2 - r^2)}{(\lambda^2 + r^2)^{\frac{n}{2}+2}}\ .
$$

It follows that

$$
\begin{aligned}
\Delta_y \left[\frac{\lambda^2 - r^2}{(\lambda^2 + r^2)^{\frac{n}{2}}} \right] \ &= \ \left[\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} \right] \left[\frac{\lambda^2 - r^2}{(\lambda^2 + r^2)^{\frac{n}{2}}} \right] \\
= \ \ & \frac{1}{(\lambda^2 + r^2)^{\frac{n}{2}+2}} \Big[-2n(\lambda^2 + r^2)^2 + n\,(n+4)\,r^2(\lambda^2 + r^2) \\
& \qquad\qquad - n^2\lambda^2(\lambda^2 + r^2) + n\,(n+2)\,r^2\,(\lambda^2 - r^2) \Big] \\
= \ \ & \frac{1}{(\lambda^2 + r^2)^{\frac{n}{2}+2}} \Big[\, n\,(n+2)\lambda^2 r^2 - n\,(n+2)\lambda^4 \Big] \qquad\qquad (\,\text{cancelation + matching}\,) \\
= \ \ & \frac{-\,n\,(n+2)\,\lambda^2\,(\lambda^2 - r^2)}{(\lambda^2 + r^2)^{\frac{n}{2}+2}}\ .
\end{aligned}
$$

Hence

$$\Delta_y \phi_o = -n(n+2)\lambda^2 \cdot \frac{(\lambda^2 - |y-\xi|^2)}{(\lambda^2 + |y-\xi|^2)^{\frac{n}{2}+2}}$$

As for

$$\begin{aligned} \Delta_y \phi_1 &= \Delta_y \left[\frac{(\xi_1 - y_1)}{(\lambda^2 + |y-\xi|^2)^{\frac{n}{2}}} \right] \\ &= 0 + 2 \, \nabla_y (\xi_1 - y_1) \cdot \nabla_y \left[\frac{1}{(\lambda^2 + |y-\xi|^2)^{\frac{n}{2}}} \right] \\ &\quad + (\xi_1 - y_1) \cdot \Delta_y \left[\frac{1}{(\lambda^2 + |y-\xi|^2)^{\frac{n}{2}}} \right] . \end{aligned}$$

$$\Delta_y \left[\frac{1}{(\lambda^2 + r^2)^{\frac{n}{2}}} \right] = \frac{2n r^2 - n^2\lambda^2}{(\lambda^2 + r^2)^{\frac{n}{2}+2}}$$

$$\begin{aligned} \Longrightarrow \quad \Delta_y \phi_1 &= \frac{2n(y_1 - \xi_1)}{(\lambda^2 + r^2)^{\frac{n}{2}+1}} + (\xi_1 - y_1) \frac{2n|y-\xi|^2 - n^2\lambda^2}{(\lambda^2 + |y-\xi|^2)^{\frac{n}{2}+2}} \\ &= \frac{-n^2\lambda^2(\xi_1 - y_1)}{(\lambda^2 + |y-\xi|^2)^{\frac{n}{2}+2}} . \end{aligned}$$

Likewise, we can find other terms in (3.8).

§ A.4. d. Proof of Corollary 3.13, Part I. Recall that

$$\|w\|^2_{\nabla} = \int_{\mathbb{R}^n} |\nabla w|^2 \,, \quad \text{and} \quad \Delta V_{\lambda,\xi} + n(n-2)\, V_{\lambda,\xi}^{\frac{n+2}{n-2}} = 0 \quad \text{in } \mathbb{R}^n .$$

Via the integration by parts formula (2.11) in Part I, we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} V_{\lambda,\,\xi}^{\frac{n+2}{n-2}}\, w &= -\frac{1}{n(n-2)} \int_{\mathbb{R}^n} (\Delta V_{\lambda,\,\xi})\, w = \frac{1}{n(n-2)} \int_{\mathbb{R}^n} \langle \nabla V_{\lambda,\,\xi} \,, \nabla w \rangle \\ &= \frac{1}{n(n-2)} \langle V_{\lambda,\,\xi} \,, w \rangle_{\nabla} = \frac{1}{n(n-2)} \langle \mathbf{z} \,, w \rangle_{\nabla} . \end{aligned}$$

Using Proposition 3.11, Part I, and the fact that $w \perp (\operatorname{span}\{\mathbf{z}\} \oplus T_{\mathbf{z}} \mathbf{Z}) \Longrightarrow \langle \mathbf{z} \,, w \rangle_{\nabla} = 0$, we obtain the inequality (3.14). Cf. Remark 4.2 in [3], pp. 50, and the proof of Lemma 2.10 (loc. cit. pp. 21).

§ A. 5. Uniform approximation.

§ A.5. a. Proof of Proposition 4.6, Part I. For $f \in (T_{\mathbf{z}}\,\mathbf{Z})^{\perp}$, write

$$f = a\,V_{\lambda,\,\xi} + \bar f \;=\; a\,\mathbf{z} \,+\, \bar f\,,$$

where $\bar f \perp (\,\text{span}\,\{\mathbf{z}\} \oplus T_{\mathbf{z}}\,\mathbf{Z}\,)$. Here $a \in \mathbb{R}$. As

$$I''_o\,(\mathbf{z})\,[f] \;=\; a\,I''_o\,(\mathbf{z})\,[\mathbf{z}] \;+\; I''_o\,(\mathbf{z})\,[\bar f\,]\,,$$

using (2.21), Part I, the Rieze Representation Theorem [cf. (2.23), Part I], and calculation similar to that in (3.15), Part I, we have

$$I''_o\,(\mathbf{z})\,[\,a\,\mathbf{z}] \;=\; -\frac{4}{n-2}\,(a\,\mathbf{z})\,.$$

Thus (4.7), Part I, holds when $\bar f \;=\; 0$. Next, we assume that $\bar f \neq 0$. Write

$$(A.5.1)\quad \bar f \;=\; \bar f_p + \bar f_o\,,\qquad \text{where}\;\; \bar f_p \;\perp\; \text{Ker}\,I''_o\,(\mathbf{z})\,[\bar f\,] \qquad \text{and}\;\; f_o \in \text{Ker}\,I''_o\,(\mathbf{z})\,[\,\bar f\,]\,.$$

Observe that

$$\text{Corollary 3.13, Part I} \quad\Longrightarrow\quad (I''_o\,(\mathbf{z})\,[\bar f\,]\,\bar f\,) \;\ge\; 2\,\bar c_2\,\|\bar f\|^2_{\triangledown} \;>\; 0$$

$$\Longrightarrow\quad \bar f \;\notin\; \text{Ker}\,I''_o\,(\mathbf{z})\,[\bar f\,] \quad\Longrightarrow\quad \bar f_p \;\neq\; 0\,.$$

Using (2.23), Part I, we obtain

$$I''_o\,(\mathbf{z})\,[\,\bar f\,] \;=\; \frac{(I''_o(\mathbf{z})\,[\bar f\,]\,\bar f_p)}{\|\bar f_p\|^2_{\triangledown}}\cdot \bar f_p \;=\; \frac{(I''_o(\mathbf{z})\,[\bar f\,]\,\bar f)}{\|\bar f_p\|^2_{\triangledown}}\cdot \bar f_p \qquad [\,\text{as}\;\;(I''_o(\mathbf{z})\,[\bar f\,]\,\bar f_o\,) \;=\; 0\,]\,.$$

Via Corollary 3.13, Part I, we have

$$\left\|\left(\frac{(I''_o(\mathbf{z})\,[\bar f\,]\,\bar f)}{\|\bar f_p\|^2_{\triangledown}}\right)\bar f_p\right\|_{\triangledown} \;\ge\; \frac{2\,\bar c_2\,\|\bar f\|^2_{\triangledown}}{\|\bar f_p\|^2_{\triangledown}}\cdot\|\bar f_p\|_{\triangledown} \;\ge\; 2\,\bar c_2\,\|\bar f\|_{\triangledown}\,,$$

as $\|\,\bar f\|_{\triangledown} \;\ge\; \|\,\bar f_p\|_{\triangledown}$. Together with (2.21), integration by parts formula (2.11) (both are found in Part I) and the fact that $\bar f \;\perp\; \mathbf{z}$, we infer that

$$I''_o\,(\mathbf{z})[\bar f\,]\,(\mathbf{z}) \;=\; 0 \;\Longrightarrow\; \mathbf{z} \in \text{Ker}\,I''_o\,(\mathbf{z})\,[\bar f\,] \;\Longrightarrow\; \mathbf{z} \;\perp\; \bar f_p \qquad [\,\text{via}\;\;(A.\,5.1)\,]\,.$$

It follows that

$$I''_o\,(\mathbf{z})\,[f] \;=\; -\left(\frac{4a}{n-2}\right)\mathbf{z} \;+\; \left(\frac{(I''_o(\mathbf{z})\,[\bar f\,]\,\bar f)}{\|\bar f_p\|^2_{\triangledown}}\right)\bar f_p$$

$$\Longrightarrow\;\; \|I''_o\,(\mathbf{z})\,[f]\,\|^2_{\triangledown} \;\ge\; \left[\left(\frac{4}{n-2}\right)^2\|\,a\,\mathbf{z}\|^2_{\triangledown} + 4\,\bar c_2^2\,\|\,\bar f\|^2_{\triangledown}\right] \;\ge\; \bar c_3^2\,\|\,f\|^2_{\triangledown}\,,$$

$$\left(\text{as}\;\;\|\,f\|^2_{\triangledown} \;=\; \|\,a\,\mathbf{z}\|^2_{\triangledown} + \|\bar f\,\|^2_{\triangledown}\right)\,.$$

Here we can take $\bar{c}_3 \;=\; \min\left\{\dfrac{4}{n-2}\,,\;\; 2\,\bar{c}_2\right\}$.

§ A.5. b. Refer to § 4 g, Part I. See also [3] (pp. 53), and [4]. As in Lemma 2.21 in [3] (pp. 27), we seek to show that the remainder

$$R_{\mathbf{z}}(w) \;:=\; I'_o(\mathbf{z}+w) \;-\; I''_o(\mathbf{z})\,[w]$$

has the uniform property

$$(A.5.2) \quad \|R_{\mathbf{z}}(w)\| \;=\; o\,(\|w\|_{\bigtriangledown}) \qquad \text{when} \quad \|\,w\|_{\bigtriangledown} \to 0 \quad (\,\text{uniform} \quad \text{in} \quad \mathbf{z} \in \mathbf{Z})\,.$$

Recall that, as operators acting on $[\bullet]$, we have

$$I'_o(\mathbf{z}+w) \;=\; \int_{\mathbb{R}^n}\Big[\,\langle\bigtriangledown(\mathbf{z}+w)\,,\bigtriangledown[\bullet]\rangle_{\bigtriangledown} \;-\; n\,(n-2)(\mathbf{z}+w)^{\frac{n+2}{n-2}}\cdot[\bullet]\,\Big]\,,$$

and
$$I''_o(\mathbf{z})[w] \;=\; \int_{\mathbb{R}^n}\Big[\,\langle\bigtriangledown w\,,\bigtriangledown[\bullet]\rangle_{\bigtriangledown} \;-\; n\,(n+2)\,\mathbf{z}^{\frac{4}{n-2}}\,w\cdot[\bullet]\,\Big]\,.$$

It follows that

(A.5.3)

$$\begin{aligned}
R_{\mathbf{z}}(w) \;&=\; \int_{\mathbb{R}^n}\Big[\,\langle\bigtriangledown z\,,\bigtriangledown[\bullet]\,\rangle_{\bigtriangledown} \;-\; n\,(n-2)(\mathbf{z}+w)_+^{\frac{n+2}{n-2}}\cdot[\bullet] \;+\; n\,(n+2)\,\mathbf{z}^{\frac{4}{n-2}}\,w\cdot[\bullet]\Big] \\
&=\; \int_{\mathbb{R}^n}\Big[-\Delta\,\mathbf{z} - n(n-2)(\mathbf{z}+w)_+^{\frac{n+2}{n-2}} + n(n+2)\,\mathbf{z}^{\frac{4}{n-2}}\,w\Big]\cdot[\bullet] \\
&=\; -n\,(n-2)\int_{\mathbb{R}^n}\Big[(\mathbf{z}+w)_+^{\frac{n+2}{n-2}} \;-\; \mathbf{z}^{\frac{n+2}{n-2}} \;-\; \frac{n+2}{n-2}\,\mathbf{z}^{\frac{4}{n-2}}\,w\Big]\cdot[\bullet]
\end{aligned}$$

$$(\,\text{applying} \quad \text{the} \quad \text{equation} \quad \Delta\,\mathbf{z} \;+\; n\,(n-2)\,\mathbf{z}^{\frac{n+2}{n-2}} \;=\; 0\,)\,.$$

One can proceed to the remark made in the proof of (ii) in Lemma 4.7 in [3] (pp. 53, that is for the subcritical case). The proof for the critical case is similar. We provide the following argument.

Using the Taylor expansion we observe that

$$(1+a)^{\frac{n+2}{n-2}} \;=\; 1 \;+\; \frac{n+2}{n-2}\cdot a \;+\; O(1)\,a^2 \qquad \text{for} \quad |\,a| \;\le\; \frac{1}{2}$$

$$\Longrightarrow \quad \left|(1+a)^{\frac{n+2}{n-2}} - 1 - \frac{n+2}{n-2}\cdot a\right| \le C a^2 \qquad \text{for} \quad |\,a| \;\le\; \frac{1}{2}\,,$$

$$(b+c)^{\frac{n+2}{n-2}} = b^{\frac{n+2}{n-2}}\left(1+\frac{c}{b}\right)^{\frac{n+2}{n-2}} = b^{\frac{n+2}{n-2}}\left[1+\frac{n+2}{n-2}\cdot\frac{c}{b} + O(1)\cdot\left(\frac{c}{b}\right)^2\right] \quad \text{for} \quad \left|\frac{c}{b}\right| \le \frac{1}{2}$$

$$\Longrightarrow \quad \left|(b+c)^{\frac{n+2}{n-2}} \;-\; b^{\frac{n+2}{n-2}} \;-\; \frac{n+2}{n-2}\,b^{\frac{4}{n-2}}\,c\right| \;\le\; C\left(\frac{c}{b}\right)b^{\frac{4}{n-2}}\cdot c \qquad \text{for} \quad \left|\frac{c}{b}\right| \;\le\; \frac{1}{2}\,.$$

Here a, b and c are numbers, whereas C is a positive constant depending on the dimension n. Given a (small) number γ, as long as

$$\left| \frac{w(y)}{\mathbf{z}(y)} \right| \ < \ \gamma \qquad\qquad (\text{recall that } \mathbf{z} \text{ is a positive function})$$

$$\Longrightarrow \quad \left| [\,\mathbf{z}(y) + w(y)]_+^{\frac{n+2}{n-2}} - [\,\mathbf{z}(y)]^{\frac{n+2}{n-2}} - \frac{n+2}{n-2}\,[\,\mathbf{z}(y)]^{\frac{4}{n-2}}\, w(y) \right| \le C\cdot\gamma\,[\,\mathbf{z}(y)]^{\frac{4}{n-2}}\, w(y)\,.$$

Let

$$\Omega_\gamma \ := \ \left\{ y \,\in\, \mathbb{R}^n \ \ | \ \ \left| \frac{w(y)}{\mathbf{z}(y)} \right| \ < \ \gamma \right\}.$$

Hence

$$\begin{aligned}
& \int_{\Omega_\gamma} \left| (\mathbf{z} + w)_+^{\frac{n+2}{n-2}} - \mathbf{z}^{\frac{n+2}{n-2}} - \frac{n+2}{n-2}\,\mathbf{z}^{\frac{4}{n-2}}\, w \right| \cdot |\,[\bullet]| \\
\le\ & C\cdot\gamma \int_{\Omega_\gamma} \mathbf{z}^{\frac{4}{n-2}}\, |\,w| \cdot |\,[\bullet]| \le C\,\gamma \int_{\mathbb{R}^n} \mathbf{z}^{\frac{4}{n-2}}\, |\,w| \cdot |\,[\bullet]| \\
\le\ & C\cdot\gamma \left(\int_{\mathbb{R}^n} |\,\mathbf{z}^{\frac{4}{n-2}}\, w|^{\frac{2n}{n+2}} \right)^{\frac{n+2}{2n}} \left(\int_{\mathbb{R}^n} [\bullet]^{\frac{2n}{n-2}} \right)^{\frac{n-2}{2n}} \\
\le\ & C_1\cdot\gamma \left(\int_{\mathbb{R}^n} |\,z|^{\frac{2n}{n-2}\cdot\frac{4}{n+2}}\, |\,w|^{\frac{2n}{n+2}} \right)^{\frac{n+2}{2n}} \ \|\,[\bullet]\|_{\bigtriangledown} \\
\le\ & C_1\cdot\gamma \left[\left(\int_{\mathbb{R}^n} |\,z|^{\frac{2n}{n-2}} \right)^{\frac{4}{n+2}} \ \left(\int_{\mathbb{R}^n} |\,w|^{\frac{2n}{n+2}\cdot\frac{n+2}{n-2}} \right)^{\frac{n-2}{n+2}} \right]^{\frac{n+2}{2n}} \ \ \|\,[\bullet]\|_{\mathcal{D}^{1\ 2}} \\
\le\ & C_2\cdot\gamma\,\|\,w\|_{\bigtriangledown}\cdot\|\,[\bullet]\|_{\bigtriangledown} \qquad\qquad\qquad (\text{applying the Sobolev inequality})\,.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& \int_{\mathbb{R}^n\setminus\Omega_\gamma} \left| (\mathbf{z} + w)_+^{\frac{n+2}{n-2}} \ - \ \mathbf{z}^{\frac{n+2}{n-2}} \ - \ \frac{n+2}{n-2}\,\mathbf{z}^{\frac{4}{n-2}}\, w \right| \cdot |\,[\bullet]| \\
\le\ & \int_{\mathbb{R}^n\setminus\Omega_\gamma} \left[(\mathbf{z} + |\,w|)_+^{\frac{n+2}{n-2}} + \mathbf{z}^{\frac{n+2}{n-2}} \ + \ \frac{n+2}{n-2}\,\mathbf{z}^{\frac{4}{n-2}}\, |\,w| \right] \cdot |\,[\bullet]| \\
\le\ & \int_{\mathbb{R}^n\setminus\Omega_\gamma} \left[(1+\gamma^{-1})^{\frac{n+2}{n-2}} |\,w|^{\frac{n+2}{n-2}} \ + \ \gamma^{-\frac{n+2}{n-2}} |\,w|^{\frac{n+2}{n-2}} \ + \ \frac{n+2}{n-2}\,\gamma^{-\frac{4}{n-2}} |\,w|^{\frac{n+2}{n-2}} \right] \cdot |\,[\bullet]| \\
& \qquad\qquad [\,|\,w(y)| \ge \gamma\cdot\mathbf{z}\,(y) \quad \text{for} \quad y \in \mathbb{R}^n\setminus\Omega_\gamma \quad \Longleftrightarrow \quad \gamma^{-1} |\,w(y)| \ge \mathbf{z}(y)\,] \\
\le\ & C_1\,\gamma^{-\frac{n+2}{n-2}} \int_{\mathbb{R}^n} |\,w|^{\frac{n+2}{n-2}} \cdot |\,[\bullet]| \\
\le\ & C_1\,\gamma^{-\frac{n+2}{n-2}} \left(\int_{\mathbb{R}^n} |\,w|^{\frac{2n}{n-2}} \right)^{\frac{n+2}{2n}} \cdot \left(\int_{\mathbb{R}^n} |\,[\bullet]|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{2n}} \\
\le\ & C_2 \left[\gamma^{-\frac{n+2}{n-2}} \cdot \|\,w\|_{\bigtriangledown}^{\frac{4}{n-2}} \right] \cdot \|\,w\|_{\bigtriangledown} \cdot \|[\bullet]\|_{\bigtriangledown}\,.
\end{aligned}$$

Thus as long as

$$(A.5.4) \qquad \gamma^{-\frac{n+2}{n-2}} \cdot \| \, w \|_{\triangledown}^{\frac{4}{n-2}} \ \le \ \gamma \quad \Longleftrightarrow \quad \| \, w \|_{\triangledown} \ \le \ \gamma^{\frac{n}{2}} \ ,$$

we have

$$(A.5.5) \ \int_{\mathbb{R}^n \setminus \Omega_\gamma} \left| (\mathbf{z} + w)_+^{\frac{n+2}{n-2}} - \mathbf{z}^{\frac{n+2}{n-2}} - \frac{n+2}{n-2} \, \mathbf{z}^{\frac{4}{n-2}} \, w \right| \cdot \| \, [\bullet] \|_{\triangledown} \le C_3 \cdot \gamma \, \| \, w \|_{\triangledown} \cdot \| \, [\bullet] \|_{\triangledown}$$

$$\Longrightarrow \quad \int_{\mathbb{R}^n} \left| (\mathbf{z} + w)_+^{\frac{n+2}{n-2}} - \mathbf{z}^{\frac{n+2}{n-2}} - \frac{n+2}{n-2} \, \mathbf{z}^{\frac{4}{n-2}} \, w \right| \cdot \| \, [\bullet] \|_{\triangledown} \le C_4 \cdot \gamma \, \| \, w \|_{\triangledown} \cdot \| \, [\bullet] \|_{\triangledown}$$

$$\Longrightarrow \quad \| \, R_{\mathbf{z}}(w) \| = \sup_{\| \, [\bullet] \|_{\triangledown} \ne 0} \frac{| \, R_{\mathbf{z}}(w)[\bullet] \, |}{\| \, [\bullet] \|_{\triangledown}} = C_5 \cdot \gamma \, \| \, w \|_{\triangledown} \, .$$

As γ can be chosen to be small [cf. (A.5.4)] when $\| \, w \|_{\triangledown} \to 0$, we obtain

$$\| \, R_{\mathbf{z}}(w) \| \ = \ o \, (1) \cdot \| \, w \|_{\triangledown} \ = \ o \, (\, \| \, w \|_{\triangledown}) \qquad \text{when} \quad \| \, w \|_{\triangledown} \to 0 \, .$$

§ A. 6. Bounded projection.

In Part I, refer to (4.11) and condition **(iii)** in Lemma 4.11 . From (2.14),

$$G \, (f) \ = \ \bar{c}_{-1} \int_{\mathbb{R}^n} H f_+^{\frac{2n}{n-2}} \qquad \text{for} \quad f \in \mathcal{D}^{1\,,\,2} \, .$$

The Fréchet derivative of G is given by

$$(A.6.1) \qquad G'(\mathbf{z} + w)([\bullet]) \ = \ - \, \bar{c}_n \int_{\mathbb{R}^n} H \cdot (\mathbf{z} + w)_+^{\frac{n+2}{n-2}} \cdot [\bullet] \qquad \text{for} \ \ [\bullet] \in \mathcal{D}^{1\,,\,2} \, .$$

It follows that

$$\begin{aligned}
(A.6.2) \quad | \, G'(\mathbf{z} + w)([\bullet]) | \ &\le \ C_1 \int_{\mathbb{R}^n} | \, \mathbf{z} + w |^{\frac{n+2}{n-2}} \, | \, [\bullet] | \qquad\qquad (\, | \, H | \ \ \text{is} \ \ \text{bounded} \,) \\
&\le \ C_1 \left(\int_{\mathbb{R}^n} | \, \mathbf{z} + w |^{\frac{n+2}{n-2} \times \frac{2n}{n+2}} \right)^{\frac{n+2}{2n}} \cdot \left(\int_{\mathbb{R}^n} | \, [\bullet] |^{\frac{2n}{n-2}} \right)^{\frac{n-2}{2n}} \\
&\le \ C_2 \left(\int_{\mathbb{R}^n} (\, \mathbf{z} + | \, w |)^{\frac{2n}{n-2}} \right)^{\frac{n+2}{2n}} \cdot \| \, [\bullet] \|_{\triangledown} \, .
\end{aligned}$$

Using the inequality

$$(\, \mathbf{z} + | \, w | \,)^{\frac{2n}{n-2}} \le C(n) \left[\, \mathbf{z}^{\frac{2n}{n-2}} + | \, w |^{\frac{2n}{n-2}} \, \right] \qquad \text{(recall that } \ \mathbf{z} \in \mathbf{Z} \ \text{ is positive),}$$

we have

$$\begin{aligned} |\, G'(\mathbf{z}+w)\,([\bullet])| \;&\le\; C_2 \left(\int_{\mathrm{I\!R}^n} |\,\mathbf{z}|^{\frac{2n}{n-2}} + |\,w|^{\frac{2n}{n-2}} \right)^{\frac{n+2}{2n}} \cdot \|\,[\bullet]\|_{\bigtriangledown} \\ &\le\; C_2 \cdot [\, C(n) + C_3]^{\frac{n+2}{2n}} \cdot \|[\bullet]\|_{\bigtriangledown} \qquad \text{for} \;\; \|\,w\|_{\bigtriangledown} \le 1\,. \end{aligned}$$

Here we use the Sobolev inequality. Therefore

$$(A.6.3) \qquad \|\, G'(\mathbf{z}+w)\| \;\le\; C \quad \text{for all} \;\; \mathbf{z} \in \mathbf{Z} \;\; \text{and} \;\; w \;\; \text{with} \;\; \|\,w\| \le 1\,.$$

§ A.7. More uniform bounds.

Refer to § 4 d, Part I. Consider the following uniform bounds.

$$(A.7.1) \qquad \begin{aligned} I_o\,(\mathbf{z}+w_\varepsilon(\mathbf{z})) \;&=\; I_o\,(\mathbf{z}) + I_o'(\mathbf{z})[w_\varepsilon(\mathbf{z})] + o\,(\|w_\varepsilon(\mathbf{z})\|) \\ &=\; \bar c_4 + o\,(\|w_\varepsilon(\mathbf{z})\|)\,; \end{aligned}$$

$$(A.7.2) \qquad G(\mathbf{z}+w_\varepsilon(\mathbf{z})) \;=\; G(\mathbf{z}) + [G'(\mathbf{z})\,|\,w_\varepsilon(\mathbf{z})] + o\,(\|\,w_\varepsilon(\mathbf{z})\|)\,;$$

$$(A.7.3) \qquad I_o'(\mathbf{z}+w_\varepsilon(\mathbf{z})) \;=\; [\,I_o''(\mathbf{z})\,|\,w_\varepsilon(\mathbf{z})] + o\,(\|w_\varepsilon(\mathbf{z})\|)\,;$$

$$(A.7.4) \qquad G'(\mathbf{z}+w_\varepsilon(\mathbf{z})) \;=\; G'(\mathbf{z}) + [G''(\mathbf{z})\,|\,w_\varepsilon(\mathbf{z})] + o\,(\|\,w_\varepsilon(\mathbf{z})\|)\,.$$

See § A .5 for (A.7.3). Here we demonstrate the arguments leading to (A.7.1) and (A.7.4) .

I. $\quad I_o\,(\mathbf{z}+w)$

$$\begin{aligned} &= \int_{\mathrm{I\!R}^n} \left(\frac{1}{2} \langle \bigtriangledown(\mathbf{z}+w)\,,\; \bigtriangledown(\mathbf{z}+w)\,\rangle - \frac{n-2}{2n} \cdot n(n-2)(\mathbf{z}+w)^{\frac{2n}{n-2}} \right) \\ &= \frac{1}{2} \int_{\mathrm{I\!R}^n} \langle \bigtriangledown \mathbf{z}\,,\; \bigtriangledown \mathbf{z}\,\rangle + \int_{\mathrm{I\!R}^n} \langle \bigtriangledown w\,,\; \bigtriangledown \mathbf{z}\,\rangle + \frac{1}{2} \int_{\mathrm{I\!R}^n} \langle \bigtriangledown w\,,\; \bigtriangledown w\,\rangle \\ &\qquad\qquad - \frac{n-2}{2n} \cdot n(n-2) \int_{\mathrm{I\!R}^n} (\mathbf{z}+w)^{\frac{2n}{n-2}} \\ &= I_o\,(\mathbf{z}) - \frac{n-2}{2n} \cdot n(n-2) \int_{\mathrm{I\!R}^n} \left[(\mathbf{z}+w)^{\frac{2n}{n-2}} - \mathbf{z}^{\frac{2n}{n-2}} - \frac{2n}{n-2} \cdot \mathbf{z}^{\frac{n+2}{n-2}} \cdot w \right] \\ &\qquad\qquad\qquad\qquad\qquad\qquad (\,\text{cf. the argument in § } A\,.5\,) \\ &= I_o\,(\mathbf{z}) + o\left(\int_{\mathrm{I\!R}^n} \mathbf{z}^{\frac{n+2}{n-2}} \cdot w \right) \qquad\qquad \left(p = \frac{2n}{n+2}\,, \quad q = \frac{2n}{n-2} \right) \\ &= I_o\,(\mathbf{z}) + o\,(\|\,w\|_{\bigtriangledown}) \qquad\qquad (\,\text{uniformly,} \quad \text{as } \|\,w\|_{\bigtriangledown} \to 0)\,. \end{aligned}$$

Here

$$
\begin{aligned}
I_o(\mathbf{z}) &= \frac{1}{2}\int_{\mathrm{I\!R}^n}\Big[\langle \nabla_o \mathbf{z}\,,\, \nabla_o \mathbf{z}\rangle \,-\, (n-2)^2\,\mathbf{z}^{\frac{2n}{n-2}}\Big] \\
&= \frac{1}{2}\int_{\mathrm{I\!R}^n}\Big[[-\Delta_o \mathbf{z}]\cdot \mathbf{z} \,-\, (n-2)^2\,\mathbf{z}^{\frac{2n}{n-2}}\Big] \\
&= \frac{1}{2}\int_{\mathrm{I\!R}^n}\Big[n\,(n-2)\,\mathbf{z}^{\frac{2n}{n-2}} \,-\, (n-2)^2\,\mathbf{z}^{\frac{2n}{n-2}}\Big] \\
&= (n-2)\int_{\mathrm{I\!R}^n}\left(\frac{1}{1+|\,y|^2}\right)^n \;=\; \bar{c}_4\,,
\end{aligned}
$$

via a translation and rescaling. Likewise,

II. $\qquad (\,G'(\mathbf{z}+w) - G'(\mathbf{z}) - G''(\mathbf{z})[\,w])\,[\bullet]$

$$
\begin{aligned}
&= C\int_{\mathrm{I\!R}^n}\left((\mathbf{z}+w)^{\frac{n+2}{n-2}} \,-\, \mathbf{z}^{\frac{n+2}{n-2}} \,-\, \frac{n+2}{n-2}\,\mathbf{z}^{\frac{4}{n-2}}\cdot w\right)[\bullet] \\
&= o\,(\|\,w\|_{\nabla}) \qquad\qquad (\,\text{uniformly,} \quad \text{as} \;\; \|\,w\|_{\nabla} \to 0)\,.
\end{aligned}
$$

Here indeed the formulas allow $w \in \mathcal{D}^{1,\,2}$ with suitable bounds on $\|\,w\|_{\nabla}$.

§ A. 8. Argument toward Proposition 4.15 in Part I.

Let $\{q_\ell^{\varepsilon\,,\,i}\}$ be an orthonormal basis of $T_{z_{\varepsilon,\,i}}\mathbf{Z}$. See (3.5). Here

$$
\mathbf{z}_{\varepsilon\,,\,i}(y) = \left(\frac{\lambda_{\varepsilon\,,\,i}}{\lambda_{\varepsilon\,,\,i}^2 + |\,y-\xi_{\varepsilon\,,\,i}|}\right)^{\frac{n-2}{2}}\,.
$$

Via the Lemma 4.1, given any $\delta > 0$, there exists a positive number $\bar{\varepsilon}$ such that

$$
(A.8.1) \qquad\qquad \|\,w_\varepsilon(\mathbf{z})\|_{\nabla} \le \delta \quad \text{for all} \;\; z \in \mathbf{Z} \;\; \text{and} \;\; |\,\varepsilon| \le \bar{\varepsilon}\,.
$$

We follow closely the argument in the proof of Theorem 2.12, as well as the notations used in [3]. As the argument is the same at every critical point $z_{\varepsilon\,,\,i}$, we simplify the notation by taking away the suffix and superfix $\varepsilon\,,\, i$. That is, in what follows

$$
\begin{aligned}
q_\ell^{\varepsilon,\,i} &\;\to\; \tilde{q}_\ell \\
\mathbf{z}_{\varepsilon,\,i} &\;\to\; \tilde{z} \\
w_{\varepsilon,\,i} &\;\to\; \tilde{w} \\
\lambda_{\varepsilon,\,i} &\;\to\; \tilde{\lambda} \\
&\text{etc.}
\end{aligned}
$$

Set

$$\mathcal{B}_{j,\,\ell} \;:=\; \tilde{\lambda}\cdot\langle D_j\, w\,,\;\; \tilde{q}_\ell\,\rangle \qquad\qquad \text{for} \;\; j\,,\;\; \ell \;=\; 0,\; 1,\; 2,\cdots,\; n\,,$$

and

$$D_o \;=\; \frac{\partial}{\partial\lambda}\,, \quad D_k \;=\; \frac{\partial}{\partial\xi_k} \qquad \text{for} \quad k \;=\; 1,\; 2,\cdots,\; n\,.$$

As $w_\varepsilon(\mathbf{z})$, the solution to the auxiliary equation, is perpendicular to $T_{\mathbf{z}}\,\mathbf{Z}\,,$ one has

$$(A.8.2) \qquad \langle\, \tilde{w}\,(\tilde{\mathbf{z}})\,,\;\tilde{q}_\ell\,\rangle_{\bigtriangledown} = 0 \qquad\qquad \text{for} \quad \ell = 0,\; 1,\; 2,\cdots,\; n$$

$$\Longrightarrow \quad \langle\, D_j\,\tilde{w}\,(\tilde{\mathbf{z}})\,,\;\tilde{q}_\ell\,\rangle_{\bigtriangledown} \,+\, \langle\,\tilde{w}(\tilde{z})\,,\; D_j\,\tilde{q}_\ell\,\rangle_{\bigtriangledown} \;=\; 0$$

$$\Longrightarrow \quad |\,\langle\, D_j\,\tilde{w}\,(\tilde{\mathbf{z}})\,,\;\tilde{q}_\ell\,\rangle_{\bigtriangledown}\,| \le \|\tilde{w}\,(\tilde{\mathbf{z}})\|_{\bigtriangledown}\cdot\|D_j\;\tilde{q}_\ell\,\|_{\bigtriangledown} \le \frac{C(n)}{\tilde{\lambda}}\cdot\|\,\tilde{w}\,(\tilde{\mathbf{z}})\|_{\bigtriangledown}$$

$$\text{(using} \quad (3.6))$$

$$\Longrightarrow \quad |\,\mathcal{B}_{j,\,\ell}| \;\le\; C(n)\cdot\|\tilde{w}\,(\tilde{\mathbf{z}})\|_{\bigtriangledown} \;\le\; C(n)\cdot\delta\,.$$

Here we apply (A.8.1). As $\tilde{\mathbf{z}}$ is a critical point of Φ_ε (see the chart in § 2 e), we have

$$(A.8.3) \qquad \langle\, I'_\varepsilon(\tilde{\mathbf{z}} + \tilde{w}\,(\tilde{\mathbf{z}}))\,, \quad D_j\,\tilde{\mathbf{z}} + D_j\,\tilde{w}\,(\tilde{\mathbf{z}})\,\rangle_{\bigtriangledown} = 0 \qquad \text{for} \;\; j = 0,\; 1,\cdots,\; n$$

$$\Longrightarrow \; \left\langle\, I'_\varepsilon(\tilde{\mathbf{z}} + \tilde{w}\,(\tilde{\mathbf{z}}))\,, \quad \frac{D_j\,\tilde{\mathbf{z}} + D_j\,\tilde{w}\,(\tilde{\mathbf{z}})}{\|D_j\,\tilde{\mathbf{z}}\|_{\bigtriangledown}}\,\right\rangle_{\bigtriangledown} \;=\; 0$$

$$\Longrightarrow \; \langle\, I'_\varepsilon(\tilde{\mathbf{z}} + \tilde{w}\,(\tilde{\mathbf{z}}))\,, \quad \tilde{q}_j\rangle_{\bigtriangledown} + \tilde{C}_j\,\tilde{\lambda}\,\left\langle\, I'_\varepsilon(\tilde{\mathbf{z}} + \tilde{w}\,(\tilde{\mathbf{z}}))\,, \quad D_j\,\tilde{w}\,(\tilde{\mathbf{z}})\right\rangle_{\bigtriangledown} \;=\; 0\,.$$

Here we use (3.2) and (3.3). Write

$$I'_\varepsilon\,(\tilde{\mathbf{z}} + \tilde{w}\,(\tilde{\mathbf{z}})) \;=\; \sum_{\ell=0}^{n} A_\ell\;\tilde{q}_\ell\,.$$

$$\text{“}\,(A.8.3)\,\text{”} \Longrightarrow \quad A_j \;+\; \sum_\ell A_\ell\,[\,\mathcal{B}_{j,\,\ell}\cdot\tilde{C}_j] \;=\; 0\,.$$

Other part of the proof follows as in the proof of Theorem 2.12 (pp. 22 in [3]) . □

Remark. The smallness of $\mathcal{B}_{j,\,\ell}$ basically said that the “tangent spaces” at $\mathbf{z}$ and at $\mathbf{z} + w(\mathbf{z})$ are almost parallel.

§ A. 9. G'.

Refer to the proof of Lemma 4.16 in Part I. Here

$$G\,(u) \;=\; -\,\bar{c}_n \frac{n-2}{2n} \int_{\mathbb{R}^n} H u_+^{\frac{2n}{n-2}} \;.$$

It follows that

$$(A.9.1)\quad G'(\mathbf{z}+w)([\bullet]) \;=\; -\,\bar{c}_n \int_{\mathbb{R}^n} H\cdot(\mathbf{z}+w)_+^{\frac{n+2}{n-2}}\cdot[\bullet] \quad \text{for } \mathbf{z}\in\mathbf{Z} \text{ and } w\in\mathcal{D}^{1,\,2}\,.$$

Hence

$$\begin{aligned}
|\,G'(\mathbf{z}+w)\,([\bullet])| \;&\le\; C_1 \int_{\mathbb{R}^n} |\,\mathbf{z}+w|^{\frac{n+2}{n-2}}\,|\,[\bullet]| \qquad (\text{as } H \text{ is bounded})\\
&\le\; C_1 \left(\int_{\mathbb{R}^n} |\,z+w|^{\frac{n+2}{n-2}\cdot\frac{2n}{n+2}} \right)^{\frac{n+2}{2n}} \left(\int_{\mathbb{R}^n} |\,[\bullet]|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{2n}}\\
&\le\; C_2 \left(\int_{\mathbb{R}^n} (\,z+|\,w|\,)^{\frac{2n}{n-2}} \right)^{\frac{n+2}{2n}} \cdot \|\,[\bullet]\|_{\bigtriangledown}\,.
\end{aligned}$$

Using the inequality

$$(\,\mathbf{z}+|\,w|\,)^{\frac{2n}{n-2}} \;\le\; C_3 \left[\, \mathbf{z}^{\frac{2n}{n-2}} \,+\, |\,w|^{\frac{2n}{n-2}} \,\right] ,$$

we obtain

$$\begin{aligned}
(A.9.2)\qquad |\,G'\,(\,\mathbf{z}+w)([\bullet])| \;&\le\; C_4 \left(\int_{\mathbb{R}^n} \mathbf{z}^{\frac{2n}{n-2}} \,+\, |\,w|^{\frac{2n}{n-2}} \right)^{\frac{n+2}{2n}} \cdot \|\,[\bullet]\|_{\bigtriangledown}\\
&\le\; C_4\,[\,C_5 \,+\, C_6]^{\frac{n+2}{2n}} \cdot \|\,[\bullet]\|_{\bigtriangledown} \qquad \text{for } \|\,w\|_{\bigtriangledown} \le 1\,.
\end{aligned}$$

Here we use the Sobolev inequality. (A.9.2) leads to

$$(A.9.3)\qquad \|\,G'\,(z+w)\| \;\le\; C \qquad \text{for all } \mathbf{z}\in\mathbf{Z} \text{ and } w \text{ with } \|\,w\|_{\bigtriangledown} \;\le\; 1\,.$$

§ A. 10. Kazdan - Warner condition.

For equation (1.1), a guiding relation is revealed when we differentiate $\mathcal{K}$ with respect to a *conformal Killing vector field* X – one that generates a family of conformal transformations. In this way we obtain the renowned Kazdan-Warner formula [8]

$$\int_{S^n} X(\mathcal{K})\, u^{\frac{2n}{n-2}}\, dV_{g_1} \;=\; 0\,, \qquad \text{where} \;\; X(\mathcal{K}) \;=\; \langle\, X\,,\; \bigtriangledown_{g_1} \mathcal{K} \rangle_{g_1}\,.$$

Simple and elegant, the Kazdan-Warner formula encapsulates a central character of the equation, namely, the *balance* between $\bigtriangledown_{g_1} \mathcal{K}$ and u.

Definition A.10.1. *A function $\hat{\mathcal{K}} \in C^1(\mathbb{R}^n)$ is said to satisfy the K-W condition if there exists a (single) positive function $f \in C^o(S^n)$ such that*

(A.10.2)

$$\int_{S^n} X(\hat{\mathcal{K}})\, f^{\frac{2n}{n-2}}\, dV_{g_1} \;=\; 0 \qquad \textit{for all conformal Killing vector fields} \quad X\,.$$

The collection of all conformal Killing vector field on S^n can be regarded as a linear space of dimension $(n+1)(n+2)/2$, with a basis formed by the generators of the dilations ($n+1$ dimension), denoted by $X_1, \cdots, X_{n+1}$, and of the rotations $[\,n(n+1)/2$ dimension$]$, denoted by $X_{n+2}\,, \cdots, X_{\frac{(n+1)(n+2)}{2}}$. Refer to [18].

Dilations. By the homogeneity of S^n, these are generated by

$$\bigtriangledown_{g_1} x_\ell \quad \text{for} \;\; \ell \;=\; 1,\; 2, \cdots,\; n+1\,.$$

Rotations. Let Θ be a rotation of S^n, which is an isometry. Consider the integral

$$\int_{S^n} [\,\hat{\mathcal{K}} \circ \Theta\,]\, f^{\frac{2n}{n-2}}\, dV_{g_1} \;=\; \int_{S^n} \hat{\mathcal{K}} \circ [\, f \circ \Theta^{-1}]^{\frac{2n}{n-2}}\, dV_{g_1}$$

Here we apply the change of variables formula, noticing that the rotation of the standard metric g_1 on S^n is isometric to itself. In particular, if

$$f(x) \;=\; \left(\frac{\lambda}{\lambda^2 + |\, y - \xi|^2} \right)^{\frac{n-2}{2}}\,,$$

then, via the Obata theorem [31], we can express

$$(A.10.3) \qquad f(\Theta^{-1}(x)) \;=\; \left(\frac{\lambda'}{\lambda'^{\,2} + |\, y - \xi'|^2} \right)^{\frac{n-2}{2}}\,.$$

§ **A.10.1.** *Pohozaev identity.* Expressing (A.10.1) in $\mathbb{R}^n$ via the stereographic projection, we obtain

$$(A.10.4) \qquad \Delta_o v + \tilde c_n K v^{\frac{n+2}{n-2}} = 0 \quad \text{in } \mathbb{R}^n,$$

where

$$(A.10.5) \qquad K(y) := \mathcal{K}(\dot{\mathcal{P}}^{-1}(y)) \qquad \text{and} \qquad v(y) = u(\dot{\mathcal{P}}^{-1}(y)) \left(\frac{2}{1+|y|^2} \right)^{\frac{n-2}{2}}.$$

[Cf. (1.5) and (2.4).] Corresponding to $X_{n+1} = \bigtriangledown_{g_1} x_{n+1}$, note that

$$\bigtriangledown_{g_1} x_{n+1} = g_1^{ij} \left[\frac{\partial [x_{n+1} \circ \dot{\mathcal{P}}^{-1}(y)]}{\partial y_i} \right] \partial_j = \frac{1}{4}(1+r^2)^2 \left[\frac{4y_i}{(1+r^2)^2} \right] \partial_i = \sum_{i=1}^n y_i \, \partial_i \,,$$

as

$$x_{n+1} = \frac{r^2 - 1}{r^2 + 1}.$$

Likewise,

$$\bigtriangledown_{g_1} \mathcal{K} = g_1^{ij} \left[\frac{\partial \mathcal{K} \circ \dot{\mathcal{P}}^{-1}(y)]}{\partial y_i} \right] \partial_j = \frac{1}{4}(1+r^2)^2 \left[\frac{\partial \mathcal{K} \circ \dot{\mathcal{P}}^{-1}(y)]}{\partial y_i} \right] \partial_i$$

Hence

$$X(\mathcal{K}) = \sum_{i=1}^n \frac{4}{1+r^2} [y_i \, \partial_i] \cdot \left[\frac{1}{4}(1+r^2)^2 \left[\frac{\partial \mathcal{K} \circ \dot{\mathcal{P}}^{-1}(y)}{\partial y_i} \right] \partial_i \right] = \sum_{i=1}^n y_i \frac{\partial \mathcal{K} \circ \dot{\mathcal{P}}^{-1}(y)}{\partial y_i}.$$

Using the stereographic transformation, (A.10.1), (A.10.4) and (A.10.5) lead to the following radial Pohozaev identities

$$(A.10.6) \qquad \int_{\mathbb{R}^n} r \frac{\partial K}{\partial r} v^{\frac{2n}{n-2}} = \sum_{i=1}^n \int_{\mathbb{R}^n} y_i \frac{\partial K}{\partial y_i} v^{\frac{2n}{n-2}} = 0 \qquad [\, K(y) = \mathcal{K} \circ \dot{\mathcal{P}}^{-1}(y)].$$

This corresponds to the rescalings by a positive number σ:

$$(A.10.7) \qquad (r, \vartheta) \mapsto (r\,\sigma, \vartheta), \quad \text{where } r \ge 0 \text{ and } \vartheta \in S^{n-1}.$$

As for $X_i = \bigtriangledown_{g_1} x_i$ when $i = 1, 2, \cdots, n$, we have (for example)

$$x_1 = \frac{2y_1}{1+r^2},$$

and

$$\bigtriangledown_{g_1} x_1 = g_1^{ij} \left[\frac{\partial [x_1 \circ \dot{\mathcal{P}}^{-1}(y)]}{\partial y_i} \right] \partial_j = \frac{1}{4}(1+r^2)^2 \left[\frac{\partial}{\partial y_i} \left(\frac{2y_1}{1+r^2} \right) \right] \partial_i$$

$$= \left[\frac{1}{2}(1+r^2) - y_1^2 \right] \partial_1 - \sum_{i=2}^n y_1 \, y_i \, \partial_i \,.$$

From (A.10.1), (A.10.4) and (A.10.5), we obtain

$$(A.10.8) \qquad \frac{1}{2}\int_{\mathbb{R}^n} \frac{\partial K}{\partial y_1}\, v^{\frac{2n}{n-2}} \;+\; \frac{1}{2}\int_{\mathbb{R}^n} r^2 \frac{\partial K}{\partial y_1}\, v^{\frac{2n}{n-2}} - \int_{\mathbb{R}^n} y_1^2\, \frac{\partial K}{\partial y_1}\, v^{\frac{2n}{n-2}}$$
$$- \sum_{i=2}^{n} \int_{\mathbb{R}^n} y_1\, y_i \frac{\partial K}{\partial y_i}\, v^{\frac{2n}{n-2}} \;=\; 0\,.$$

It follows that if we have the translational Pohozaev identity:

$$(A.10.9) \qquad \int_{\mathbb{R}^n} \frac{\partial K}{\partial y_1}\, v^{\frac{2n}{n-2}} = 0\,,$$

$$\text{then} \qquad (A.10.8) \implies \frac{1}{2}\int_{\mathbb{R}^n} r^2 \frac{\partial K}{\partial y_1}\, v^{\frac{2n}{n-2}} - \sum_{i=1}^{n} \int_{\mathbb{R}^n} y_1\, y_i \frac{\partial \hat{K}}{\partial y_i}\, v^{\frac{2n}{n-2}} \;=\; 0$$

$$\implies \int_{\mathbb{R}^n} r^2 \frac{\partial K}{\partial y_1}\, v^{\frac{2n}{n-2}} \;=\; 2\int_{\mathbb{R}^n} y_1 \left[\sum_{i=1}^{n} y_i \frac{\partial K}{\partial y_i}\right] v^{\frac{2n}{n-2}}$$

$$(A.10.10) \cdots\cdots \qquad \implies \int_{\mathbb{R}^n} r^2 \frac{\partial K}{\partial y_1}\, v^{\frac{2n}{n-2}} \;=\; 2\int_{\mathbb{R}^n} y_1 \left[r\frac{\partial K}{\partial r}\right] v^{\frac{2n}{n-2}}\,.$$

Theorem A.10.11. *For a given* $\mathcal{H} \in C^1(S^n)$, *let* $H(y) = \mathcal{H}\,(\dot{\mathcal{P}}^{-1}(y))$. *Suppose* $G_{|\mathbf{z}}$ [*corresponding to* H] *has a critical point. Then*

$$\hat{\mathcal{K}} \;=\; 1 + \varepsilon\,\mathcal{H}$$

fulfills the K-W *condition* (A.10.2) *for any* $\varepsilon \in \mathbb{R}$.

Proof. Let $(\lambda_c\,,\ \xi_c)$ be a critical point of $G_{|\mathbf{z}}$. It follows from formulas (6.1) and (6.3) that

$$(A.10.12) \quad \frac{\partial G_{|\mathbf{z}}}{\partial \lambda}(\lambda_c\,,\ \xi_c) \;=\; 0 \implies n\int_{\mathbb{R}^n} H(y)\, \frac{\lambda_c^{n-1}}{(\,\lambda_c^2 \,+\, |\,y-\xi_c|^2)^n}$$
$$-2n\int_{\mathbb{R}^n} H(y)\, \frac{\lambda_c^{n+1}}{(\,\lambda_c^2 + |\,y-\xi_c|^2)^{n+1}} \;=\; 0\,,$$

$$(A.10.13) \quad \frac{\partial G_{|\mathbf{z}}}{\partial \xi_i}(\lambda_c\,,\ \xi_c) \;=\; 0 \implies \int_{\mathbb{R}^n} H(y)\, \frac{\lambda_c^n\,(\xi_{ci} - y_i)}{(\,\lambda_c^2 + |\,y-\xi_c|^2)^{n+1}} \;=\; 0\,.$$

Here $H(y) = \mathcal{H}\,(\dot{\mathcal{P}}^{-1}(y))$.

Translational Case. From (A.10.13) we have

$$(A.10.14) \qquad \int_{\mathbb{R}^n} H(y)\, \frac{\partial}{\partial y_i} \left[\left(\frac{\lambda_c}{\lambda_c^2 + |\, y - \xi_c|^2} \right)^n \right] = 0\,.$$

We have

$$|\, H| \le C_1 \quad \text{and} \quad |\bigtriangledown H(y)| \le \frac{C_2}{|\, y|^2} \quad \text{for} \quad |\, y| \gg 1$$

(see Lemma 9.37 in [24]). Also,

$$\lim_{|\, y_1| \to \infty} \left[H\,(y_1\,, \cdots,\ y_n) \cdot \left(\frac{\lambda_c}{\lambda_c^2 + |\, y - \xi_c|^2} \right)^n \right] = 0 \quad \text{for} \quad (y_2\,, \cdots,\ y_n) \in \mathbb{R}^{n-1}.$$

Via Fubini's theorem and integration by parts, we obtain from (A.10.13) the translational Pohozaev identity:

$$(A.10.15) \qquad \int_{\mathbb{R}^n} \frac{\partial H}{\partial y_1}\, w^{\frac{2n}{n-2}} = 0$$

with

$$(A.10.16) \qquad w(y) = \left(\frac{\lambda_c}{\lambda_c^2 + |\, y - \xi_c|^2} \right)^{\frac{n-2}{2}}.$$

Radial Case. Consider the integral

(A.10.17)

$$\begin{aligned}
&\sum_{i=1}^n \int_{\mathbb{R}^n} H(y)\, \frac{\partial}{\partial y_i} \left[(y_i - \xi_{ci}) \cdot \left(\frac{\lambda_c}{\lambda_c^2 + |\, y - \xi_c|^2} \right)^n \right] \\
&= \sum_{i=1}^n \int_{\mathbb{R}^n} H(y) \left[\left(\frac{\lambda_c}{\lambda_c^2 + |\, y - \xi_c|^2} \right)^n - \frac{2n\,(y_i - \xi_{ci})^2 \lambda_c^n}{[\,\lambda_c^2 + |\, y - \xi_c|^2]^{n+1}} \right] \\
&= n \int_{\mathbb{R}^n} H(y) \left(\frac{\lambda_c}{\lambda_c^2 + |\, y - \xi_c|^2} \right)^n - 2n \int_{\mathbb{R}^n} H(y)\, \frac{|\, y - \xi_c|^2 \lambda_c^n}{[\,\lambda_c^2 + |\, y - \xi_c|^2]^{n+1}} \\
&= n \int_{\mathbb{R}^n} H(y) \left(\frac{\lambda_c}{\lambda_c^2 + |\, y - \xi_c|^2} \right)^n - 2n \int_{\mathbb{R}^n} H(y)\, \frac{[|\, y - \xi_c|^2 + \lambda_c^2 - \lambda_c^2]\, \lambda_c^n}{[\,\lambda_c^2 + |\, y - \xi_c|^2]^{n+1}} \\
&= -n \int_{\mathbb{R}^n} H(y) \left(\frac{\lambda_c}{\lambda_c^2 + |\, y - \xi_c|^2} \right)^n + 2n \int_{\mathbb{R}^n} H(y)\, \frac{\lambda_c^{n+2}}{[\,\lambda_c^2 + |\, y - \xi_c|^2]^{n+1}} \\
&= 0 \qquad\qquad [\,\text{using} \quad (\text{A.10.12})]\,.
\end{aligned}$$

On the other hand, via Fubini's theorem and integration by parts, we obtain

$$\begin{aligned}
&\sum_{i=1}^n \int_{\mathbb{R}^n} H(y)\,\frac{\partial}{\partial y_i}\left[(y_i-\xi_{ci})\cdot\left(\frac{\lambda_c}{\lambda_c^2+|\,y-\xi_c|^2}\right)^n\right]\\
= \ & -\sum_{i=1}^n \int_{\mathbb{R}^n} \frac{\partial H}{\partial y_i}\left[(y_i-\xi_{ci})\cdot\left(\frac{\lambda_c}{\lambda_c^2+|\,y-\xi_c|^2}\right)^n\right]\\
= \ & -\sum_{i=1}^n \int_{\mathbb{R}^n} y_i\,\frac{\partial H}{\partial y_i}\cdot\left(\frac{\lambda_c}{\lambda_c^2+|\,y-\xi_c|^2}\right)^n+\sum \xi_{ci}\int_{\mathbb{R}^n}\frac{\partial H}{\partial y_i}\cdot\left(\frac{\lambda_c}{\lambda_c^2+|\,y-\xi_c|^2}\right)^n\\
= \ & -\int_{\mathbb{R}^n}\left(\sum_{i=1}^n y_i\,\frac{\partial H}{\partial y_i}\right)\cdot\left(\frac{\lambda_c}{\lambda_c^2+|\,y-\xi_c|^2}\right)^n \qquad\qquad \text{(using (A.10.15)}\\
= \ & -\int_{\mathbb{R}^n} r\,\frac{\partial H}{\partial r}\cdot\left(\frac{\lambda_c}{\lambda_c^2+|\,y-\xi_c|^2}\right)^n = 0 \qquad\qquad \text{(via } (A.10.17))\,.
\end{aligned}$$

Cf. the radial Pohozaev identity (A.10.6). Running the argument backward as in (A.10.6), we have

(A.10.18)

$$\int_{S^n} X_{n+1}\,(\mathcal{H})\,f^{\frac{2n}{n-2}}\,dV_{g_1} \ = \ 0\,, \quad \text{where } f(x) \ = \ \left(\frac{\lambda_c}{\lambda_c^2+|\,y-\xi_c|^2}\right)^{\frac{n-2}{2}}\cdot\left(\frac{1+|\,y|^2}{2}\right)^{\frac{n-2}{2}}.$$

Cf. also (A.10.5) .

$X_1\,,\cdots\ X_n$ *directions.* From (A.10.7) , (A.10.8), (A.10.9) and (A.10.10), we need to show that

$$(A.10.19)\quad \int_{\mathbb{R}^n} r^2\,\frac{\partial \hat{H}}{\partial y_1}\left(\frac{\lambda_c}{\lambda_c^2+|\,y-\xi_c|^2}\right)^n \ = \ 2\int_{\mathbb{R}^n} y_1\left[r\frac{\partial \hat{H}}{\partial r}\right]\left(\frac{\lambda_c}{\lambda_c^2+|\,y-\xi_c|^2}\right)^n$$

in order to obtain

(A.10.20)

$$\int_{S^n} X_i\,(\mathcal{H})\,f^{\frac{2n}{n-2}}\,dV_{g_1} \ = \ 0\,, \quad \text{for } i \ = \ 1,\ 2,\ \cdots,\ n\,, \ \text{where } f(x) \text{ is given in (A.10.18)}\,.$$

Using Fubini's theorem and integration by parts, we obtain

$$\begin{aligned}
(A.10.21)\quad & \int_{\mathbb{R}^n} r^2\,\frac{\partial H}{\partial y_1}\left(\frac{\lambda_c}{\lambda_c^2+|\,y-\xi_c|^2}\right)^n\\
= \ & \int_{\mathbb{R}^n} H\frac{\partial}{\partial y_1}\left[\,r^2\left(\frac{\lambda_c}{\lambda_c^2+|\,y-\xi_c|^2}\right)^n\right]\\
= \ & -2\int_{\mathbb{R}^n} y_1\,H\left(\frac{\lambda_c}{\lambda_c^2+|\,y-\xi_c|^2}\right)^n dy \ + \ 2n\int_{\mathbb{R}^n} r^2 H\cdot\frac{(y_1-\xi_{c1})\,\lambda_c^n}{(\lambda_c^2+|\,y-\xi_c|^2)^{n+1}}\,dy\,.
\end{aligned}$$

On the other side,

$$(A.10.22) \qquad 2\int_{\mathbb{R}^n} y_1 \left[r\frac{\partial H}{\partial r}\right] \left(\frac{\lambda_c}{\lambda_c^2 + |\,y-\xi_c|^{\,2}}\right)^n$$

$$= \; 2\int_{S^{n-1}}\int_0^\infty \frac{\partial H}{\partial r}\left[r^n y_1 \left(\frac{\lambda_c}{\lambda_c^2 + |\,y-\xi_c|^{\,2}}\right)^n \right] dr\, dS_\vartheta$$

$$= \; -2n\int_{S^{n-1}}\int_0^\infty H\, y_1\, r^{n-1}\left(\frac{\lambda_c}{\lambda_c^2 + |\,y-\xi_c|^{\,2}}\right)^n dr\, dS_\vartheta$$

$$-\,2\int_{S^{n-1}}\int_0^\infty H\, y_1\, r^{n-1}\left(\frac{\lambda_c}{\lambda_c^2 + |\,y-\xi_c|^{\,2}}\right)^n dr\, dS_\vartheta$$

$$+\,2n\int_{S^{n-1}}\int_0^\infty H\, y_1\, r^{n}\left(\frac{\lambda_c^n\left(2r - 2\sum \xi_{c_i}\cdot \frac{y_i}{r}\right)}{[\,\lambda_c^2 + r^{\,2} - 2\sum \xi_{c_i}\cdot y_i + |\,\xi_c|^{\,2}\,]^{n+1}}\right) dr\, dS_\vartheta\,,$$

where we apply integration by parts formula.

The last term in (A.10.22)

$$= \; 2n\int_{S^{n-1}}\int_0^\infty H\, y_1\, r^{n-1}\left(\frac{\lambda_c^n\,(\,2r^{\,2} - 2\sum \xi_{c_i}\cdot y_i)}{(\,\lambda_c^2 + |\,y-\xi_c|^{\,2})^{n+1}}\right) dr\, dS_\vartheta$$

$$= \; 2n\int_{S^{n-1}}\int_0^\infty H\, y_1\, r^{n-1}\left[\frac{\lambda_c^n\cdot(\,\lambda_c^2 + r^{\,2} - 2\sum \xi_{c_i}\cdot y_i + |\,\xi_c|^{\,2} + r^{\,2} - |\,\xi_c|^{\,2} - \lambda_c^2)}{(\,\lambda_c^2 + |\,y-\xi_c|^{\,2})^{n+1}}\right] dr\, dS_\vartheta$$

$$= \; 2n\int_{S^{n-1}}\int_0^\infty H\, y_1\, r^{n-1}\left(\frac{\lambda_c}{\lambda_c^2 + |\,y-\xi_c|^{\,2}}\right)^n dr\, dS_\vartheta$$

$$+\,2n\int_{S^{n-1}}\int_0^\infty H\, y_1\, r^{\,2}\cdot r^{n-1}\left(\frac{\lambda_c^n}{(\lambda_c^2 + |\,y-\xi_c|^{\,2})^{n+1}}\right) dr\, dS_\vartheta$$

$$-2n\,(|\,\xi_c|^{\,2} + \lambda_c^2)\int_{S^{n-1}}\int_0^\infty H\, y_1\,\cdot r^{n-1}\left(\frac{\lambda_c^n}{(\lambda_c^2 + |\,y-\xi_c|^{\,2})^{n+1}}\right) dr\, dS_\vartheta\,.$$

Hence

$$(A.10.23) \qquad 2\int_{\mathbb{R}^n} y_1 \left[r\frac{\partial H}{\partial r}\right] \left(\frac{\lambda_c}{\lambda_c^2 + |\,y-\xi_c|^{\,2}}\right)^n$$

$$= \; -\,2\int_{S^{n-1}}\int_0^\infty H\, y_1\, r^{n-1}\left(\frac{\lambda_c}{\lambda_c^2 + |\,y-\xi_c|^{\,2}}\right)^n dr\, dS_\vartheta$$

$$+2n\int_{S^{n-1}}\int_0^\infty H\, r^{\,2}\cdot r^{n-1} y_1\left(\frac{\lambda_c^n}{(\lambda_c^2 + |\,y-\xi_c|^{\,2})^{\,n+1}}\right) dr\, dS_\vartheta$$

$$-2n\,(|\,\xi_c|^{\,2}+\lambda_c^2)\int_{S^{n-1}}\int_0^\infty H\,y_1\,r^{n-1}\left(\frac{\lambda_c^n}{(\lambda_c^2+|\,y-\xi_c|^{\,2})^{\,n+1}}\right)dr\,dS_\vartheta$$

$$\begin{aligned}
= \quad & -2\int_{\mathbb{R}^n} H\,y_1\left(\frac{\lambda_c}{\lambda_c^2+|\,y-\xi_c|^{\,2}}\right)^n \\
& +2n\int_{\mathbb{R}^n} H\,r^{\,2}(y_1-\xi_{c1})\left(\frac{\lambda_c^n}{[\,\lambda_c^2+|\,y-\xi_c|^{\,2}]^{\,n+1}}\right) \\
& \quad -2n\,(|\,\xi_c|^{\,2}+\lambda_c^2)\int_{\mathbb{R}^n} H\cdot(y_1-\xi_{c1})\left(\frac{\lambda_c^n}{[\,\lambda_c^2+|\,y-\xi_c|^{\,2}]^{\,n+1}}\right) \\
& +2n\int_{\mathbb{R}^n} H\,r^{\,2}\,\xi_{c1}\left(\frac{\lambda_c^n}{[\,\lambda_c^2+|\,y-\xi_c|^{\,2}]^{n+1}}\right) \\
& \quad -2n\,(|\,\xi_c|^{\,2}+\lambda_c^2)\int_{\mathbb{R}^n} H\,\xi_{c1}\left(\frac{\lambda_c^n}{[\,\lambda_c^2+|\,y-\xi_c|^{\,2}]^{\,n+1}}\right)\,.
\end{aligned}$$

The third term above is equal to zero via (A.10.13). The first two terms cancel with the two terms in (A.10.21). Thus we are required to show the sum of the last two terms in (A.10.23) is equal to zero.

(A.10.24)

The sum of the last two terms in (A.10.23)

$$\begin{aligned}
= \quad & 2n\,\xi_{c1}\int_{\mathbb{R}^n} H\left[\,r^{\,2}-|\,\xi_c|^{\,2}-\lambda_c^2\right]\left(\frac{\lambda_c^n}{(\lambda_c^2+|\,y-\xi_c|^{\,2})^{n+1}}\right) \\
= \quad & 2n\,\xi_{c1}\int_{\mathbb{R}^n} H\left[\,(\lambda_c^2+|\,y-\xi_c|^{\,2})-2\,\lambda_c^2-2\,|\,\xi_c|^{\,2}+2\sum y_i\cdot\xi_{ci}\right]\left(\frac{\lambda_c^n}{(\lambda_c^2+|\,y-\xi_c|^{\,2})^{n+1}}\right) \\
= \quad & 2n\,\xi_{c1}\,\lambda_c\left[\int_{\mathbb{R}^n} H\,\frac{\lambda_c^{n-1}}{(\lambda_c^2+|\,y-\xi_c|^{\,2})^n}\;-\;2\int_{\mathbb{R}^n} H\,\frac{\lambda_c^{n+1}}{(\lambda_c^2+|\,y-\xi_c|^{\,2})^{n+1}}\;\right] \\
& +2n\,\xi_{c1}\left[\int_{\mathbb{R}^n} H\,\frac{\lambda_c^n[\,2\,\sum y_i\cdot\xi_{ci}]}{(\lambda_c^2+|\,y-\xi_c|^{\,2})^{n+1}}\;-\;\int_{\mathbb{R}^n} H\,\frac{\lambda_c^n\,[\,2|\,\xi_c|^{\,2}]}{(\lambda_c^2+|\,y-\xi_c|^{\,2})^{n+1}}\;\right] \\
= \quad & 2n\,\xi_{c1}\int_{\mathbb{R}^n} H\,\frac{\lambda_c^n[\,2\,\sum(y_i-\xi_{ci})\cdot\xi_{ci}]}{(\lambda_c^2+|\,y-\xi_c|^{\,2})^{n+1}}\;=\;2n\,\xi_{c1}\sum\xi_{ci}\int_{\mathbb{R}^n} H\,\frac{\lambda_c^n[\,y_i-\xi_{ci}]}{(\lambda_c^2+|\,y-\xi_c|^{\,2})^{n+1}} \\
= \quad & 0\,.
\end{aligned}$$

Here we use (A.10.13). Combining (A.10.21) – (A.10.24), we obtain (A.10.20).

Rotations. Let Θ_t be a family of rotations so that

$$\frac{d\Theta_t}{dt}\bigg|_{\,t\,=\,0} \;=\; X_k \qquad \text{for} \quad k\in\mathbb{N} \;\;\text{with}\;\; = \;n+2\;\le\;k\;\le\;(n+1)(n+2)/2\,.$$

As in (A.10.4), we have

$$\int_{S^n} X_k(K)\, f^{\frac{n+2}{n-2}} dV_{g_1} \;=\; \frac{d}{dt}\int_{S^n} \mathcal{K}(\Theta_t)\, f^{\frac{n+2}{n-2}}\, dV_{g_1}\Big|_{t=0} \;=\; \frac{d}{dt}\int_{S^n} K\,[\,f(\Theta_t^{-1})]^{\frac{n+2}{n-2}}\, dV_{g_1}\Big|_{t=0}$$

$$= \frac{d}{dt}\int_{\mathbb{R}^n} H\left(\frac{\lambda_t}{\lambda_t^2 + |y - |\,\xi_t|^2}\right)^n \Big|_{t=0} \;=\; 0$$

$$(\lambda_o \;=\; \lambda_c\,, \quad \xi_o \;=\; \xi_c)$$

via the chain rule and (A.10.12) and (A.10.13). □

Here we point out that, in the special case $(\,\lambda_c,\ \xi_c) \;=\; (1,\ \vec{0}\,)$,

$$\text{“(6.5) \& (6.6) (Part I)”} \Longrightarrow \int_{S^n} \mathcal{H}(x)\, x_\ell\, dS_x \;=\; 0$$

$$\Longrightarrow \int_{S^n} \mathcal{H}(x)\, \Delta_1\,(x_\ell)\, dS_x \;=\; 0 \quad \Longrightarrow \int_{S^n} \langle \bigtriangledown \mathcal{H}(x)\,,\ \bigtriangledown_{g_1}(x_\ell)\rangle\, dS_x \;=\; 0$$

$$\Longrightarrow \int_{S^n} X_\ell\,(\mathcal{H})\, dS_x \;=\; 0$$

for $\ell \;=\; 0,\ 1, \cdots,\ n$. Here $X_\ell \;=\; \bigtriangledown_{g_1} x_\ell$ is the conformal Killing vector field on S^n generating the dilations (see [18]). Observe that when $(\,\lambda_c\,,\ \xi_c) \;=\; (1\,,\ \vec{0}\,)$, $\psi \equiv 1$ as described in Theorem 6.10 (Part I).

We conclude this section with the following (well-known) remark.

Proposition A.10.25. *Suppose $\mathcal{K} \in C^1(S^n)$ satisfy the K-W condition and $\Phi\,:\,S^n \to S^n$ is a conformal transformation. then $\mathcal{K}\circ\Phi$ also satisfies the K-W condition.*

Proof. We first note that Φ induces an isomorphism on the collection of all conformal Killing vector field. Suppose $\phi_t\ :\ S^n \to S^n$ is a parameter of conformal transformations which generate X. That is,

$$\frac{d\,\phi_t}{dt}\Big|_{t=0} \;=\; X\,.$$

This results

$$\int_{S^n} X(\mathcal{K})\, f^{\frac{2n}{n-2}}\, dV_{g_1} = \left[\frac{d}{dt}\int_{S^n} [\mathcal{K}\circ\phi_t\,]\, f^{\frac{2n}{n-2}}\, dV_{g_1}\right]_{t=0}.$$

Then

$$\int_{S^n} X(\mathcal{K}\circ\,\Phi)\, f^{\frac{2n}{n-2}}\, dV_{g_1} \;=\; \left[\frac{d}{dt}\int_{S^n} [\mathcal{K}\circ\Phi\circ\phi_t\,]\, f^{\frac{2n}{n-2}}\, dV_{g_1}\right]_{t=0}$$

$$= \int_{S^n} \tilde{X}(\mathcal{K}\circ\,\Phi)\, f^{\frac{2n}{n-2}}\, dV_{g_1} = 0\,,$$

where

$$\tilde{X} \;=\; \frac{d\,[\,\Phi \circ \phi_t\,]}{dt}\Big|_{\,t\,=\,0}\,.$$

Hence $\mathcal{K} \circ \Phi$ also satisfies the K-W condition. $\square$

§ A. 11. Deriving (7.14) and (7.15) , Part I.

Following (7.3) and (7.4) in Part I , a direct calculation shows that

$$\begin{aligned}
\frac{\partial^2 G_{|\mathbf{z}}}{\partial \lambda^2}\,(1,\;\vec{0}\,) \;&=\; -2\,n\,(n+2)\,\bar{c}_{-1} \int_{\mathbb{R}^n} \frac{H(\,y)}{(\,1+|\,y|^{\,2})^{n+1}} \\
&\qquad + 4n\,(n+1)\,\bar{c}_{-1} \int_{\mathbb{R}^n} \frac{H(\,y)}{(\,1+|\,y|^{\,2})^{n+2}} \\
&=\; -\,\frac{n\,(n+2)}{2^n}\,\bar{c}_{-1} \int_{S^n} (1-x_{n+1})\,H(x)\,dS_x \\
&\qquad + \frac{n\,(n+1)}{2^n}\,\bar{c}_{-1} \int_{S^n} (1-x_{n+1})^2\,H(x)\,dS_x \\
&=\; \frac{n\,(n+1)}{2^n}\,\bar{c}_{-1}\left[\int_{S^n} x_{n+1}^2\,H(x)\,dS_x - \frac{1}{n+1}\int_{S^n} H(x)\,dS_x\right] \\
&\qquad\qquad \left(\text{using} \int_{S^n} x_{n+1}\,H(x)\,dS_x \;=\; 0\right)\,,
\end{aligned}$$

In addition,

$$\begin{aligned}
\frac{\partial^2 G_{|\mathbf{z}}}{\partial \xi_\ell^2}\,(1,\;\vec{0}\,) \;&=\; -\,2n\,\bar{c}_{-1} \int_{\mathbb{R}^n} \frac{H(\,y)}{(\,1+|\,y|^{\,2})^{n+1}} \\
&\qquad + 4n\,(n+1)\,\bar{c}_{-1} \int_{\mathbb{R}^n} H(\,y) \frac{(y_\ell)^2}{(\,1+|\,y|^{\,2})^{n+2}} \\
&= -\,\frac{n}{2^n}\,\bar{c}_{-1} \int_{S^n} (1\,-\,x_{n+1})\,H(x)\,dS_x \;+\; \frac{n\,(n+1)}{2^n}\,\bar{c}_{-1} \int_{S^n} (x_\ell)^2\,H(x)\,dS_x \\
&= -\,\frac{n}{2^n}\,\bar{c}_{-1} \int_{S^n} H(x)\,dS_x \;+\; \frac{n\,(n+1)}{2^n}\,\bar{c}_{-1} \int_{S^n} (x_\ell)^2\,H(x)\,dS_x\,.
\end{aligned}$$

Here we use

$$x_{n+1} \;=\; \frac{r^2-1}{r^2+1} \quad\Longrightarrow\quad \frac{1}{1+r^2} \;=\; \frac{1-x_{n+1}}{2}\,.$$

§ A. 12. $\lambda_M = \sqrt{(t+\Delta)(t-\Delta)}$ is the only critical point in expression (2.12), Part II.

Refer to paragraph proceeding the proof of Lemma 2.21, Part II. The argument is based on the following.

(i) The function $\dfrac{a}{a^2+\lambda^2}$ is decreasing on λ ; it is decreasing in a when $a > \lambda$.

(ii) $\dfrac{t+\Delta}{\lambda} > \dfrac{t-\Delta}{\lambda} \implies \bar{\phi}_+ > \bar{\phi}_-$. See (2.13) in Part II .

We divide the argument into three parts.

(a) When $\lambda < t-\Delta$. We have

$$\frac{t+\Delta}{\lambda} > \frac{t-\Delta}{\lambda} > 1 \implies \pi > \bar{\phi}_+ > \bar{\phi}_- > \frac{\pi}{2} \implies \sin\bar{\phi}_+ < \sin\bar{\phi}_- .$$

Thus we have

$$\lambda < t-\Delta \implies \quad -\underset{\uparrow\ \text{(smaller)}}{\frac{2\,[\,t+\Delta]}{[\,t+\Delta]^2+\lambda^2}}\,[\sin\bar{\phi}_+]^{n-1} + \underset{\uparrow\ \text{(smaller)}}{\frac{2\,[\,t-\Delta]}{[\,t-\Delta]^2+\lambda^2}}\,[\sin\bar{\phi}_-]^{n-1} > 0$$

$$\implies \quad \frac{\partial}{\partial\lambda}\int_{2\arctan\left(\frac{t-\Delta}{\lambda}\right)}^{2\arctan\left(\frac{t+\Delta}{\lambda}\right)} [\sin\varphi]^{n-1}\,d\varphi > 0 \qquad [\text{via } (2.13)\,, \text{ Part II}]\,.$$

(b) Likewise,

$$\lambda > t+\Delta \implies \quad -\underset{\uparrow\ \text{(bigger)}}{\frac{2\,[\,t+\Delta]}{[\,t+\Delta]^2+\lambda^2}}\,[\sin\bar{\phi}_+]^{n-1} + \underset{\uparrow\ \text{(bigger)}}{\frac{2\,[\,t-\Delta]}{[\,t-\Delta]^2+\lambda^2}}\,[\sin\bar{\phi}_-]^{n-1} < 0$$

$$\implies \quad \frac{\partial}{\partial\lambda}\int_{2\arctan\left(\frac{t-\Delta}{\lambda}\right)}^{2\arctan\left(\frac{t+\Delta}{\lambda}\right)} [\sin\varphi]^{n-1}\,d\varphi < 0 \qquad [\text{via } (2.13)\,, \text{ Part II}]\,.$$

(c) *Cross over*. When $t+\Delta > \lambda > t-\Delta$. We have

$$0 < \bar{\phi}_- < \frac{\pi}{2} \quad \text{and} \quad \frac{\pi}{2} < \bar{\phi}_+ < \pi\,.$$

Thus

$$\sin\bar{\phi}_- \downarrow \text{ from } 1 \quad \text{and} \quad \sin\bar{\phi}_+ \uparrow \text{ to } 1 \quad \text{when } \lambda \text{ increases in this range}\,.$$

While

$$\frac{[\,t+\Delta]}{[\,t+\Delta]^{\,2}+\lambda^{\,2}} \;<\; \frac{[\,t-\Delta]}{[\,t-\Delta]^{\,2}+\lambda^{\,2}} \qquad \text{when} \;\; \lambda \;=\; t-\Delta$$

$$\Longrightarrow \qquad \frac{\partial}{\partial\lambda}\int_{2\arctan\left(\frac{t-\Delta}{\lambda}\right)}^{2\arctan\left(\frac{t+\Delta}{\lambda}\right)} [\sin\varphi]^{n-1}\,d\varphi \;<\; 0$$

$$[\,\text{via}\;\;(2.13)\,,\;\text{Part}\;\;\text{II}\,]\,.$$

$$\frac{[\,t+\Delta]}{[\,t+\Delta]^{\,2}\;+\;\lambda^{\,2}} \;>\; \frac{[\,t-\Delta]}{[\,t-\Delta]^{\,2}+\lambda^{\,2}} \qquad \text{when} \;\; \lambda \;=\; t+\Delta$$

$$\Longrightarrow \qquad \frac{\partial}{\partial\lambda}\int_{2\arctan\left(\frac{t-\Delta}{\lambda}\right)}^{2\arctan\left(\frac{t+\Delta}{\lambda}\right)} [\sin\varphi]^{n-1}\,d\varphi \;>\; 0 \qquad [\,\text{via}\;\;(2.13)\,,\;\text{Part}\;\;\text{II}\,]\,.$$

Moreover, $\left[\dfrac{[\,t\,+\,\Delta]}{[\,t\,+\,\Delta]^{\,2}+\lambda^{\,2}} \;-\; \dfrac{[\,t\,-\,\Delta]}{[\,t\,-\,\Delta]^{\,2}\,+\,\lambda^{\,2}}\right] \uparrow\;\text{in}\;\lambda\;\uparrow\;(>0)\,.$

Thus we see that there can only be one critical point in $(\,t\,-\,\Delta\,,\;t\,+\,\Delta)\,.$

§ A. 13. Deriving (2.39) in Part II.

From (2.38), Part II, we have

$$\begin{aligned}
(2\varrho)^2 \;&=\; \left[\frac{2(R_\lambda+\delta_\lambda)}{(R_\lambda+\delta_\lambda)^2+1} \;-\; \left(-\,\frac{2(R_\lambda-\delta_\lambda)}{(R_\lambda\,-\,\delta_\lambda)^2+1}\right)\right]^2 \\
&\qquad\qquad +\;\left[\frac{(R_\lambda+\delta_\lambda)^2-1}{(R_\lambda+\delta_\lambda)^2+1} \;-\; \frac{(R_\lambda-\delta_\lambda)^2-1}{(R_\lambda-\delta_\lambda)^2+1}\right]^2 \\
&=\; \frac{1}{[(R_\lambda\,+\,\delta_\lambda)^2\,+\,1]^2\,[(R_\lambda\,-\,\delta_\lambda)^2\,+\,1]^2}\,\times \\
&\quad \times\Big(\Big\{2(R_\lambda+\delta_\lambda)[(R_\lambda\,-\,\delta_\lambda)^2\,+\,1]\;+\;2(\,R_\lambda\,-\,\delta_\lambda)[\,(R_\lambda\,+\,\delta_\lambda)^2\;+\;1]\Big\}^2 \\
&\qquad\qquad +\,[\,(\,\mathcal{A}-1)(\,\mathcal{B}+1)-(\,\mathcal{B}-1)(\,\mathcal{A}+1)]^2\Big) \\
&=\; \frac{4\,[\,2R_\lambda+(R_\lambda+\delta_\lambda)(R_\lambda-\delta_\lambda)(2R_\lambda)]^2\;+\;[\,2(\,\mathcal{A}-\,\mathcal{B})]^2}{[(R_\lambda+\delta_\lambda)^2+1]^2\,[(R_\lambda-\delta_\lambda)^2+1]^2} \\
&=\; \frac{4\cdot(\;[2R_\lambda(1+R_\lambda^2-\delta_\lambda^2)]^{\,2}\;+\;[(R_\lambda+\delta_\lambda)^2-(R_\lambda-\delta_\lambda)^2]^{\,2})}{[(R_\lambda+\delta_\lambda)^2+1]^2\,[(R_\lambda-\delta_\lambda)^2+1]^2}\,.
\end{aligned}$$

In the above,

$$\mathcal{A} \;=\; (R_\lambda\,+\,\delta_\lambda)^2 \quad\;\; \text{and} \quad\;\; \mathcal{B} \;=\; (R_\lambda\,-\,\delta_\lambda)^2.$$

It follows that

$$(2\varrho)^2 \;=\; \frac{(4R_\lambda)^2\,[\,(1+R_\lambda^2-\delta_\lambda^2)^2 \;+\; 4\,\delta_\lambda^2\,]}{[(R_\lambda+\delta_\lambda)^2\,+\,1]^2\;[(R_\lambda-\delta_\lambda)^2\,+\,1]^2}\,.$$

§A. 14. Back to S^n (refer to § 3 j in Part II) .

Lemma A.14.1. *Given a positive function* $\mathcal{K} \in C^{1,\,\alpha}\,(S^n)\,,$ *let* $K(y) = \mathcal{K} \circ \dot{\mathcal{P}}^{-1}(y)$ *be defined for* $y \in \mathbb{R}^n$. *Suppose* $v \in \mathcal{D}^{1,\,2} \cap C^2(\mathbb{R}^n)$ *is a positive solution of*

$$\Delta_o\, v \;+\; (\tilde{c}_n\, K)\, v^{\frac{n+2}{n-2}} \;=\; 0 \qquad in \quad \mathbb{R}^n.$$

Set

$$(A.14.2) \qquad u\,(x) \;=\; v\,(y)\cdot\left(\frac{1+|y|^{\,2}}{2}\right)^{\frac{n-2}{2}} \qquad for \;\; x \;=\; \dot{\mathcal{P}}^{-1}(y) \in S^n \setminus \{\mathbf{N}\}\,.$$

Then u has a removable singularity at north pole $\mathbf{N}$, *and satisfies the equation*

$$(A.14.3) \qquad \Delta_1\, u -\; \tilde{c}_n\, n\,(n-1) + (\tilde{c}_n\,\mathcal{K})\, u^{\frac{n+2}{n-2}} \;=\; 0 \quad in \;\; S^n.$$

Proof. As $v \in \mathcal{D}^{1\,2}$, via the Sobolev inequality (see for instance (2.9) in Part I [25]) , there exists a positive number C such that

$$(A.14.4) \qquad \int_{|y|\ge 1} |\,\bigtriangledown v(y)|^{\,2} \;+\; \int_{|y|\ge 1} v^{\frac{2n}{n-2}}\,(y) \;\le C\,.$$

By using the Kelvin transform

$$\tilde{y} \;=\; \frac{y}{|y|^{\,2}} \qquad \text{for} \quad |\,y| \;\ge\; 1\,,$$

we let

$$(A.14.5) \qquad \tilde{v}\,(\tilde{y}) \;=\; v(y)\cdot\frac{1}{|\tilde{y}|^{n-2}} \qquad \text{with} \quad \tilde{y} \;=\; \frac{y}{|y|^{\,2}}\,.$$

The equation

$$\Delta_o\, v(y) \;+\; (\tilde{c}_n\, K)\, v^{\frac{n+2}{n-2}}(y) \;=\; 0 \quad \text{ for } \;\; |\,\tilde{y}| \ge 1$$

is transformed into the equation

$$\Delta_o\, \tilde{v}\,(\tilde{y}) + \left[\tilde{c}_n\cdot K\,\left(\frac{\tilde{y}}{|\tilde{y}|^{\,2}}\right)\right]\, \tilde{v}^{\frac{n+2}{n-2}}(\tilde{y}) \;=\; 0 \quad \text{ for } \;\; 0 \,<\, |\,y| \,\le\, 1\,.$$

Furthermore, from (A.14.4), there exists a positive number C_1 such that

$$\int_{|\tilde{y}|\le 1} |\bigtriangledown \tilde{v}\,(\tilde{y})|^2\, d\tilde{y} \;+\; \int_{|\tilde{y}|\,\le\,1} \tilde{v}^{\frac{2n}{n-2}}\,(\tilde{y})\, d\tilde{y} \;\le\; C_1\,.$$

By a result of Brezis and Kato [11], we have $\tilde{v} \,\in\, L^\infty(B_o(\frac{1}{2}))$. It follows that

$$v\,(y) \;\le\; \frac{C_2}{|\,y|^{n-2}} \qquad \text{for} \quad |\,y| \gg 1\,.$$

That is, u is bounded in a deleted neighborhood of $\mathbf{N}$. Standard elliptic theory can be applied to show that u has a removable singularity at $\mathbf{N}$, and via the standard conformal transform (A.14.2), we show that equation (A.14.3) is fulfilled. □

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Construction of Blow-up Sequences for the Prescribed Scalar Curvature Equation on S^n. I. Uniform Cancelation

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Abstract

For $n \ge 6$, using the Lyapunov - Schmidt reduction method, we describe how to construct (scalar curvature) functions on S^n, so that each of them enables the conformal scalar curvature equation to have an infinite number of positive solutions, which form a blow-up sequence. The prescribed scalar curvature function is shown to have $C^{\,n-1,\,\beta}$ smoothness. We present the argument in two parts. In this first part, we discuss the uniform cancelation property in the Lyapunov - Schmidt reduction method for the scalar curvature equation. We also explore relation between the Kazdan-Warner condition and the first order derivatives of the reduced functional, and symmetry in the second order derivatives of the reduced functional.

Key Words: Scalar Curvature Equation; Blow-up; Critical Points; Sobolev Spaces.

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§ 1. Introduction.

We apply the Lyapunov - Schmidt reduction method to find non-constant function $\mathcal{K}$ so that the equation

$$(1.1)\qquad \Delta_1 u - \tilde{c}_n n (n-1) u + (\tilde{c}_n \mathcal{K}) u^{\frac{n+2}{n-2}} = 0 \quad \text{in} \quad S^n$$

has infinite number of positive solutions $\{u_i\}_{i=1}^{\infty}$, which compose a blow-up sequence of solutions. We refer to § 1 c for the (rather standard) notations we use.

The blow - up phenomenon is endogenous to equation (1.1), because of the critical Sobolev exponent linked to $(n+2)/(n-2)$ (cf. § 2 b.1). It is systematically studied (see for examples [11] [12] [13] [19] [21] [22] [24] [25] [27]). When $\mathcal{K}$ equals to a positive constant, say, $(\tilde{c}_n \mathcal{K}) \equiv n(n-2)$, equation (1.1) has a family of positive solutions (via the stereographic projection $\dot{\mathcal{P}}$ which sends the north pole $\mathbf{N}$ to ∞):

$$\left\{ U_{\lambda,\ \xi}(x) := \left(\frac{\lambda}{\lambda^2 + |\,y - \xi|^{\,2}} \right)^{\frac{n-2}{2}} \cdot \left(\frac{1 + |\,y|^2}{2} \right)^{\frac{n-2}{2}} \qquad \text{with} \ \ (\lambda,\ \xi) \in \mathbb{R}^+ \times \mathbb{R}^n \right\}.$$

Here $y = \dot{\mathcal{P}}(x)$ for $x \in S^n \setminus \mathbf{N}$, and $U_{\lambda,\ \xi}(\mathbf{N}) := [\lambda/2]^{\frac{n-2}{2}}$. In particular, any sequence $\{U_{\lambda_i,\ \xi_i}\}_{i=1}^{\infty}$ with $\lambda_i \to 0^+$ and $\xi_i \to \vec{0}$ is a blow - up sequence. The simple idea on perturbing $\{U_{\lambda,\ \xi}\}$ to form a blow - up sequence encounters a remarkable obstacle on stabilizing and fixing the scalar curvature function $\mathcal{K}$ (non- constant this time).

A significant development comes about when Ambrosetti, Berti and Malchiodi, using the Lyapunov - Schmidt reduction method, construct many C^k - metrics g on S^n such that the Yamabe equation

$$(1.2)\qquad \Delta_g u - (\tilde{c}_n R_g) u + \tilde{c}_n n (n-1) u^{\frac{n+2}{n-2}} = 0 \quad \text{in} \quad S^n$$

has infinite number of positive solutions [here g is non - conformal to g_1 – the standard metric on S^n; R_g is the scalar curvature of $(S^n,\ g)$]. See the renowned monograph [4]. This project stems from a study on these wonderful results and seek to apply them to the scalar curvature equation (1.1) so as to construct blow - up sequences.

As we work through the Lyapunov - Schmidt method, we encounter two of the main features of equation (1.1), namely, the balance and flexibility (refer to § 6 a and § 7). The key object is the collection of solutions

$$(1.3)\quad \mathbf{Z} := \left\{ V_{\lambda,\ \xi} \in \mathcal{D}^{1,\,2} \;\middle|\; V_{\lambda,\ \xi}(y) := \left(\frac{\lambda}{\lambda^2 + |\,y - \xi|^{\,2}} \right)^{\frac{n-2}{2}} \ \text{with} \ \ (\lambda,\ \xi) \in \mathbb{R}^+ \times \mathbb{R}^n \right\}$$

of the equation

$$(1.4)\qquad \Delta_o v_o + n(n-2) v_o^{\frac{n+2}{n-2}} = 0 \quad \text{in} \quad \mathbb{R}^n.$$

'Sitting' inside the Hilbert space $\mathcal{D}^{1,2}$ [see (2.7) for the definition], the pulled-back metric on $\mathbf{Z}$ is isometric (up to a rescaling) to the hyperbolic metric on the upper half space $\mathbb{R}^+ \times \mathbb{R}^n$ (this is shown in § 3 b). We find it curious that $\mathbf{Z}$ naturally carries such a structure. This fact, although not directly linked to the main aim of the paper, provides valuable intuition on the vast amount of space for the solutions to be separated (see § 5 a), and the correct order to be expected in the derivatives (see § 2 in [23], in particular, Lemma 2.65 in [23]).

Via the stereographic projection $\dot{\mathcal{P}}$ (see § 2 a for more detail), equation (1.1) is transformed into

$$\Delta_o v + (\tilde{c}_n K) v^{\frac{n+2}{n-2}} = 0 \qquad \text{in} \quad \mathbb{R}^n . \tag{1.5}$$

See (2.4) and (2.6) for the relations between u and v, and between $\mathcal{K}$ and K. Let us write

$$K = 4n(n-1) + \varepsilon H. \tag{1.6}$$

Note that $\tilde{c}_n \cdot [\, 4n(n-1)\,] = n(n-2)$. Consider the 'interaction' of H with $\mathbf{Z}$:

$$G_{|\mathbf{Z}}(\mathbf{z}) = \bar{c}_{-1} \int_{\mathbb{R}^n} H(y) \left[\frac{\lambda}{\lambda^2 + |\, y - \xi\,|^2} \right]^n \qquad \text{for} \;\; \mathbf{z} = V_{\lambda,\, \xi} \in \mathbf{Z}. \tag{1.7}$$

Here $\bar{c}_{-1}$ is a *negative* constant defined in (2.15). For $\varepsilon \in \mathbb{R}$ small, the question on finding a positive solution of equation (1.5) is reduced to finding a *stable* critical point of $G_{|\mathbf{Z}}$. See [4]. This is discussed in § 2 f.

The balance condition is expressed by the existence of a critical point for $G_{|\mathbf{Z}}$, and is linked to the Kazdan-Warner condition (§ 6 c). To study the flexibility part, we show that the Hessian matrix of $G_{|\mathbf{Z}}$ at a critical point is always trace-free, thus can never be positive or negative definite. Instead of looking for strict local maximum or minimum (as in the Yamabe equation), we seek *saddle points*. The contrast here is that one cannot 'catch' a saddle point by looking at the values along the boundary and comparing them to the value of an inside point (cf. the proofs of Theorem 2.16 in [4] and Proposition 24 in [8]).

In [2], using the Lyapunov-Schimdt method on the scalar curvature equation (1.1), (finite number of) stable critical points of the $G_{|\mathbf{Z}}$ are found by using a degree counting method. The research is carried on in [3], where the authors explore symmetries (see also § 7 c). Each of these methods does not readily juxtapose to produce a blow-up sequence of solutions. In this article and its sequel [23], we introduce annular domains (see § 1 a), and determine precisely the critical points of the *reduced functional* $G_{|\mathbf{Z}}$, and show that the Hessian matrix at the critical point is non-degenerate (hence a stable critical point). By superimposing concentric annular domains, and carefully estimating

the gradient interference, we are able to find infinite number of stable critical points via degree theory for maps.

§ 1 a. *Description of the main result.* Consider the sequence of annular domains

$$B_o\left(\frac{1+\eta}{\mathbf{a}}\right) \Big\backslash \overline{B_o\left(\frac{1-\eta}{\mathbf{a}}\right)}\,, \quad \cdot\ \cdot\ \cdot\,, \quad B_o\left(\frac{1+\eta}{\mathbf{a}^k}\right) \Big\backslash \overline{B_o\left(\frac{1-\eta}{\mathbf{a}^k}\right)}\,, \quad \cdot\ \cdot\ \cdot\ .$$

We describe the numbers $\mathbf{a} > 1$ and $\eta \in (0,\ 1)$ in (1.9). Choose a small positive number σ and fix a number $\tau \in (n-1\,,\ n)$. Let H^S be given by

$$\text{(1.8)} \qquad H^S(y) \;=\; \begin{cases} \dfrac{1}{\mathbf{a}^{\tau k}} & \text{if} \quad y \in B_o\left(\dfrac{1+\eta}{\mathbf{a}^k}\right) \Big\backslash B_o\left(\dfrac{1-\eta}{\mathbf{a}^k}\right), \qquad k = 1,\ 2, \cdots, \\[2ex] 0 & \text{if} \quad y \not\in \bigcup_{k=1}^{\infty} \left\{ B_o\left(\dfrac{1+(\eta+\sigma)}{\mathbf{a}^k}\right) \Big\backslash B_o\left(\dfrac{1-(\eta+\sigma)}{\mathbf{a}^k}\right) \right\}. \end{cases}$$

In between $B_o\left(\frac{1+(\eta+\sigma)}{\mathbf{a}^k}\right) \backslash B_o\left(\frac{1+\eta}{\mathbf{a}^k}\right)$ and $B_o\left(\frac{1-\eta}{\mathbf{a}^k}\right) \backslash B_o\left(\frac{1-(\eta+\sigma)}{\mathbf{a}^k}\right)$, we properly smooth out H^S so that $H^S \in C^{\,n-1,\ \beta}(\mathbb{R}^n)$ (not necessarily in a rotationally symmetric manner, see § 3 g in Part II [23]). Together with Part II [23], We show the following result.

Main Theorem. *For* $n \ge 6$ *, let* $H^S \in C^{\,n-1,\ \beta}(\mathbb{R}^n)$ *be as described in* (1.8)*, with the parameter* τ *and* η *satisfying*

$$\text{(1.9)} \qquad \tau \in (n-1\,,\ n)\,, \quad 1 - A^2 \;>\; \eta \;>\; B^2 \;>\; 0 \quad \textit{and} \quad \frac{1+\eta}{1-\eta} \;\le\; \frac{5}{2}\,.$$

Here A *and* B *are positive numbers. There exist positive constants* C*,* C_1*,* c *and* ε_o *so that if the parameters* $\mathbf{a}$ *and* σ *in* (1.8) *satisfy*

$$\mathbf{a} \;>\; C^2 \quad \textit{and} \quad 0 \;<\; \sigma \;<\; c^2\,,$$

then the equation

$$\Delta_o\, v + \tilde{c}_n\left[\,4n\,(n-1) \,+\, \varepsilon\, H^S\,\right] v^{\frac{n+2}{n-2}} \;=\; 0 \qquad \textit{in} \quad \mathbb{R}^n$$

has an infinite number of positive solutions $\{v_m\}_{m=1}^{\infty} \subset C^{\,2,\ \bar\beta}(\mathbb{R}^n)$ *whenever* $|\varepsilon| \le \varepsilon_o$*. Moreover,*

$$\text{(1.10)} \qquad \|\, v_m - V_{\lambda_m\,,\ \xi_m}\|_{\bigtriangledown} \;\le\; C_1 \cdot \varepsilon \qquad \textit{for} \quad m = 1,\ 2, \cdots.$$

Here $\lambda_m \to 0$ *and* $|\,\xi_m| \to 0$ *as* $m \to \infty$*. As a result,* 0 *is a blow-up point for* $\{v_m\}_{m=1}^{\infty}$*.*

In (1.10), $\|\ \ \|_{\bigtriangledown}$ represents the L^2-norm on gradient for the Hilbert space $\mathcal{D}^{1,\ 2}$. See (2.7). We also demonstrate in [23] how to transfer these solutions $\{v_i\}_{i=1}^{\infty}$ back to S^n as solutions of equation (1.1). [In (1.9), the number 5/2 appears naturally in the calculations. It is, however, not claimed to be sharp.]

In this first part, we discuss the uniform cancelation property in the Lyapunov-Schmidt reduction method for equation (1.1). This uniformity enables us to find an infinite number of solutions. Already used in the Yamabe problem, and mentioned in [4], this property is known by experts working on the area. Nevertheless, it is an advantage to have the details recorded down. This is done in § 4. Given the available references, we select those (relatively) routine arguments and put them in the e-Appendix, which can be downloaded at

www.math.nus.edu.sg/˜matlmc/e-Appendix.pdf

§ 1 b. *Additional highlights of the article.*

$*_1$ In § 6, we present the first derivatives of the reduced functional $G_{|\mathbf{z}}$, and provide geometric interpretations of the formulas. The discussion culminates in showing that the existence of a critical point for $G_{|\mathbf{z}}$ implies the fulfillment of the Kazdan-Warner condition (refer to Theorem 6.10 for full detail).

$*_2$ In § 7, we apply the formulas on second derivatives of $G_{|\mathbf{z}}$ to obtain existence result [for a single solution of equation (1.1)] under symmetry conditions. See Theorem 7.18.

§ 1 c. *Conventions.* Throughout this work, we assume that the dimension $n \ge 3$, except when otherwise is specifically mentioned, and let $\tilde{c}_n = (n-2)/[\,4\,(n-1)\,]$. We observe the practice of using C, possibly with sub-indices, to denote various positive constants, which may be rendered *differently* from line to line according to the contents. *Whilst we use $\bar{c}$ and $\bar{C}$, possibly with sub-index, to denote a* fixed *positive constant which always keeps the same value as it is first defined*. [The *negative* constant $\bar{c}_{-1}$ is defined in (2.15).]

$\bullet_1$ Denote by $B_y(r)$ the open ball in $\mathbb{R}^n$ with center at y and radius $r > 0$, and $\| S^n \|$ the measure of S^n in $\mathbb{R}^{n+1}$ with respect to the standard metric.

$\bullet_2$ Δ_g is the Laplace-Beltrami operator associated with the metric g on S^n. Likewise, Δ_o is the Laplace-Beltrami operator associated with the Euclidean metric g_o on $\mathbb{R}^n$, and Δ_1 is the Laplace-Beltrami operator associated with the standard metric g_1 on S^n.

$\bullet_3$ Whenever there is no risk of misunderstanding, we suppress dy from the integral expressions.

§ 2. The Lyapunov - Schmidt reduction : the case of the conformal scalar curvature equation on $\mathbb{R}^n$.

§ 2 a. *Stereographic projection onto* $\mathbb{R}^n$. Consider the stereographic projection

$$
\begin{aligned}
(2.1) \qquad \dot{\mathcal{P}} \,:\, S^n \setminus \{\mathbf{N}\} \;&\longrightarrow\; \mathbb{R}^n \\
x \;&\mapsto\; y = \dot{\mathcal{P}}(x)\,, \quad \text{where } \; y_i \;=\; \frac{x_i}{1 - x_{n+1}}\,, \quad 1 \le i \le n\,.
\end{aligned}
$$

Here $x = (x_1\,, \cdots,\, x_{n+1}) \in S^n \setminus \{\mathbf{N}\} \subset \mathbb{R}^{n+1}$, and $\mathbf{N} = (0,\, \cdots,\, 0,\, 1)$. Conversely,

$$
(2.2) \qquad x_i \;=\; \frac{2\,y_i}{1+r^2}\,, \qquad 1 \le i \le n\,, \qquad \text{and} \quad x_{n+1} \;=\; \frac{r^2-1}{r^2+1}\,, \qquad \text{where } \; r \;=\; |\,y\,|\,.
$$

It is known that $\dot{\mathcal{P}}$ is a conformal map between $(S^n \setminus \{\mathbf{N}\}\,,\ g_1)$ and $(\mathbb{R}^n,\ g_o)$. The conformal factor is given by

$$
(2.3) \qquad g_1\,(x) \;=\; \left[\frac{4}{(1+r^2)^2}\right] g_o\,(y) \qquad\qquad \text{for } \; y \;=\; \dot{\mathcal{P}}(x)\,.
$$

Set

$$
(2.4) \qquad v\,(y) \;:=\; u\,(\dot{\mathcal{P}}^{-1}(y)) \cdot \left(\frac{2}{1+|y|^2}\right)^{\frac{n-2}{2}} \qquad \text{for } \; y \in \mathbb{R}^n.
$$

Suppose u is a positive solution of equation (1.1). Then we have

$$
(2.5) \qquad \Delta_o\, v + (\tilde{c}_n\, K)\, v^{\frac{n+2}{n-2}} \;=\; 0 \qquad \text{in } \; \mathbb{R}^n\,.
$$

We denote

$$
\mathcal{K} \;=\; 4n\,(n-1) + \varepsilon\,\mathcal{H}\,.
$$

In (2.5), for $y \in \mathbb{R}^n$, we set

$$
(2.6) \quad K(y) \;:=\; \mathcal{K}\,(\dot{\mathcal{P}}^{-1}(y)) \;=\; 4n\,(n-1) \,+\, \varepsilon\, H(y) \quad \text{where } \; H\,(y) \;:=\; \mathcal{H}\,(\dot{\mathcal{P}}^{-1}(y))\,.
$$

§ 2 b. *The Hilbert space and the (non-linear) functional.* As there is an excellent book [4] on the topic, we follow closely [4] and set up the basic notations for later study. Throughout this article we work on the Hilbert space

$$
(2.7) \quad \mathcal{D}^{1,\,2} \;=\; \mathcal{D}^{1,\,2}(\mathbb{R}^n) \;:=\; \left\{ f \in L^{\frac{2n}{n-2}}(\mathbb{R}^n) \bigcap W^{1,\,2}_{\text{loc}}\,(\mathbb{R}^n) \;\middle|\; \int_{\mathbb{R}^n} \langle \nabla f\,,\, \nabla f\rangle < \infty \right\}.
$$

The inner product is defined by

$$
(2.8) \qquad \langle\, f\,,\ \psi\,\rangle_{\nabla} \;:=\; \int_{\mathbb{R}^n} \langle \nabla f\,,\, \nabla \psi\rangle \quad \text{for} \quad f\,,\ \psi \in \mathcal{D}^{1,\,2}\,,
$$

which induces the norm denoted by $\| \ \|_{\bigtriangledown}$. We recall the following properties.

§ 2 b.1. *Sobolev's embedding with critical exponent.* (See, for instance, [8].) There exists a dimensional constant $\bar{\mathcal{S}}_n$ (see [1]) such that

$$(2.9) \qquad \left(\int_{\mathbb{R}^n} |\, f|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{2n}} \ \le \ \bar{\mathcal{S}}_n \left(\int_{\mathbb{R}^n} \| \bigtriangledown f \|^{\,2} \right)^{\frac{1}{2}} \qquad \text{for all} \quad f \in \mathcal{D}^{1,\,2}.$$

§ 2.b.2. *Integration by parts formula.* The following condition is sufficient for our use here. Let $h \in \mathcal{D}^{1,\,2} \cap C^2(\mathbb{R}^n)$ with

$$(2.10) \qquad \int_{\mathbb{R}^n} |\, \Delta_o\, h|^{\frac{2n}{n+2}} \ < \ \infty\,.$$

Then we have

$$(2.11) \qquad \int_{\mathbb{R}^n} \langle \bigtriangledown f\,,\, \bigtriangledown h \rangle \ = \ - \int_{\mathbb{R}^n} f \cdot (\Delta_o\, h) \qquad \text{for all} \quad f \ \in \ \mathcal{D}^{1,\,2}.$$

This can be shown by using Sobolev inequality (2.9), and the fact that $\mathcal{D}^{1,\,2}$ coincides with the completion of $C_o^\infty(\mathbb{R}^n)$ with respect to the L^2-norm of the gradient [2]. See § A.1 in the e-Appendix.

§ 2.b.3. *The non-linear functional.* Corresponding to equation (1.5) with (1.6), consider the functional $I_\varepsilon \colon \mathcal{D}^{1,\,2} \to \mathbb{R}$ given by

(2.12)

$$I_\varepsilon(f) \ := \ \int_{\mathbb{R}^n} \left[\frac{1}{2} \langle \bigtriangledown f\,,\, \bigtriangledown f \,\rangle \ - \ \frac{n-2}{2n} \cdot n\,(n-2)\, f_+^{\frac{2n}{n-2}} \right] \ - \ \varepsilon \cdot \frac{n-2}{2n} \int_{\mathbb{R}^n} (\,\tilde{c}_n\, H\,)\, f_+^{\frac{2n}{n-2}}\ .$$

Here f_+ denotes the positive part of f . Separately, we define

$$(2.13) \qquad I_o\,(f) = \frac{1}{2} \int_{\mathbb{R}^n} \left[\langle \bigtriangledown_o f\,,\, \bigtriangledown_o f \rangle - (n-2)^2\, f_+^{\frac{2n}{n-2}} \right] \qquad \text{for} \ \ f \in \mathcal{D}^{1,\,2}\,,$$

$$(2.14) \qquad G\,(H)\,(f) \ = \ \bar{c}_{-1} \int_{\mathbb{R}^n} H\, f_+^{\frac{2n}{n-2}} \qquad \text{for} \ \ f \in \mathcal{D}^{1,\,2} \ \ \Longrightarrow \ \ I_\varepsilon \ = \ I_o + G(H)\,.$$

Here

$$(2.15) \qquad \bar{c}_{-1} \ := \ -\,\frac{n\,-\,2}{2\,n} \cdot \tilde{c}_n \ = \ -\,\frac{(n\,-\,2)^2}{8n\,(n-1)}\ .$$

(The subindex of a negative sign is to remind ourselves that the constant is negative.) In most cases there is no risk of confusion, we simply write

$$G\,(H)\,(f) \ = \ G\,(f)\,.$$

Given $\mathcal{K} \in C^\beta(S^n)$ for $\beta \in (0,\ 1)$, (2.6) implies that H is bounded and $H \in C^\beta(\mathbb{R}^n)$. It can be shown that any critical point v of the functional I_ε satisfies equation (2.1). Moreover, standard elliptic theory and maximum principle imply that $v \in C^2(\mathbb{R}^n)$ and $v > 0$. See [17] and [28].

§ 2 c. *First order information – I'_o and its kernel.* For $f \in \mathcal{D}^{1,2}$, a calculation using (2.13) shows that the Fréchet derivative of I_o is given by

$$(2.16) \qquad I'_o(f)\,[\,h] \;=\; \int_{\mathbb{R}^n} \Big[\langle \nabla f \,,\, \nabla h \rangle - n(n-2)\, f_+^{\frac{n+2}{n-2}} \cdot h \Big] \qquad \text{for} \;\; h \in \mathcal{D}^{1,\,2}.$$

We know that (see [4]), for $v_o \in \mathcal{D}^{1,\,2}$,

$$I'_o\,(v_o) \;\equiv\; 0 \quad \Longrightarrow \quad v_o > 0 \;\; \text{is} \;\; \text{smooth} \;\; \text{and} \;\; \Delta\, v_o + n\,(n-2)\, v_o^{\frac{n+2}{n+2}} \;=\; 0 \quad \text{in} \;\; \mathbb{R}^n.$$

The classification theorem obtained by Caffarelli-Gidas-Spruck/Gidas-Ni-Nirenberg [6] [15] [16] shows that if v_o is as above, then

$$(2.17) \qquad v_o\,(y) \;=\; V_{\lambda,\,\xi}\,(y) \;:=\; \left(\frac{\lambda}{\lambda^2 + |\,y - \xi|^2} \right)^{\frac{n-2}{2}} \qquad \text{for} \quad y \in \mathbb{R}^n.$$

Here $\xi \in \mathbb{R}^n$ is fixed and $\lambda > 0$. As set out in (1.3),

$$(2.18) \qquad \mathbf{Z} \;=\; \text{Ker}\; I'_o\,.$$

With the parametrization

$$(2.19) \qquad \begin{array}{rcl} \Phi \,:\, \mathbb{R}^+ \times \mathbb{R}^n & \to & \mathbf{Z} \\ (\,\lambda\,,\,\xi) & \mapsto & V_{\lambda,\,\xi}\,, \end{array}$$

$\mathbf{Z}$ is a $(n+1)$ - dimension submanifold in $\mathcal{D}^{1,2}$.

§ 2 d. *Second order information – I''_o and its kernel.* Here we treat the second Fréchet derivative of I_o at $f \in \mathcal{D}^{1,\,2}$ as a bilinear map on $\mathcal{D}^{1,\,2}$. See pp. 23 in [5]. The expression is given by

$$(2.20) \quad (I''_o(f)\,[\,\phi]\; h) \;=\; \int_{\mathbb{R}^n} \Big[\langle \nabla \phi\,,\, \nabla h \rangle - n\,(n+2)\, f_+^{\frac{4}{n-2}}\, \phi \cdot h \,\Big] \quad \text{for} \;\; \phi\,, \;\; h \in \mathcal{D}^{1,\,2}\,.$$

In particular, for $\mathbf{z} \in \mathbf{Z}$,

$$(2.21) \quad (I''_o(\mathbf{z})\,[\,\phi]\; h) \;=\; \int_{\mathbb{R}^n} \Big[\langle \nabla \phi\,,\, \nabla h \rangle - n\,(n+2)\, \mathbf{z}^{\frac{4}{n-2}}\, \phi \cdot h \,\Big] \quad \text{for} \;\; \phi\,, \;\; h \in \mathcal{D}^{1,\,2}\,.$$

Thus the condition

$$(I''_o(\mathbf{z})\,[\,\phi]\; h) \;\equiv\; 0 \qquad \text{for} \quad \text{all} \quad h \in \mathcal{D}^{1,2}$$

implies that

$$(2.22) \qquad \Delta_o \phi \;+\; n\,(n+2)\, \mathbf{z}^{\frac{4}{n-2}}\, \phi \;=\; 0 \quad \text{in} \;\; \mathbb{R}^n.$$

Equation (2.22) is also studied in [4] [19].

Using Riesz Representation Theorem (see, for example, § A.2 in the e-Appendix), we identify

$$(2.23) \qquad I_o''(\mathbf{z}) \,:\, \mathcal{D}^{1,\,2} \;\to\; \mathcal{D}^{1,\,2}$$

$$f \;\mapsto\; \frac{(I_o''(\mathbf{z})\,[f]\;h)}{\|h\|_{\bigtriangledown}^2}\cdot h \qquad \{\text{here } \; h \not\equiv 0\,; \quad h\;\perp\,\mathrm{Ker}\,I_o''(\mathbf{z})\,[f]\,\}\,.$$

For the reduction method to work, we need the following matching on the kernel of $I_o''(\mathbf{z})$ (refer to [4]).

Lemma 2.24. *For any* $\mathbf{z}\in\mathbf{Z}$, $I_o''(\mathbf{z})$, *as rendered in* (2.23), *is an index* 0 *Fredholm operator. Moreover,*

$$\mathrm{Ker}\,I_o''(\mathbf{z}) \;=\; T_{\mathbf{z}}\,\mathbf{Z} \;\subset\; \mathcal{D}^{1,\,2}\,.$$

Let

$$(2.25) \qquad \mathbf{W}_{\mathbf{z}} \;:=\; [\,\mathrm{Ker}\;I_o''(\mathbf{z})\,]^{\perp} \;=\; [\,T_{\mathbf{z}}\,\mathbf{Z}]^{\perp} \;\subset\; \mathcal{D}^{1,\,2}\,.$$

Then

$$I_o''(\mathbf{z}) \,:\, \mathbf{W}_{\mathbf{z}} \to \mathbf{W}_{\mathbf{z}}$$

is an isomorphism.

See the proof of Lemma 2.10 in pp. 21 ([4]), and Lemma 5.2 in pp. 61 ([4]). See also § A.2 in the e-Appendix. Cf. [12] and Lemma 4.2 in [19]. We first observe that, in (2.23) and with the notations used there,

$$(2.26) \qquad \mathrm{Ker}\,I_o''(\mathbf{z}) \;=\; T_{\mathbf{z}}\,\mathbf{Z} \quad\Longrightarrow\quad T_{\mathbf{z}}\,\mathbf{Z} \subset \mathrm{Ker}\,I_o''(\mathbf{z})\,[f]$$

$$\Longrightarrow \quad h\,\in\,\mathbf{W}_{\mathbf{z}} \qquad\qquad (\,\text{as } \; h\perp\mathrm{Ker}\,I_o''(\mathbf{z})\,[f]\,)\,.$$

§ 2 e. *The flow chart.* The simplicity of the Lyapunov-Schmidt reduction method is exposed in the book [4]. We keep the notations close to those used in [4]. We use the following chart to present the set-up of the Lyapunov-Schmidt reduction method, and encourage readers to refer to the excellent book [4].

$$
\begin{array}{ll}
I_\varepsilon(f) \;=\; I_o(f) \;+\; \varepsilon\, G(f) & \textit{for } \; f \in \mathcal{D}^{1,\,2}\,. \\
\downarrow & \\
\mathbf{Z} \;=\; \operatorname{Ker} I'_o & \textit{Refer to } (2.18)\,. \\
\downarrow & \\
T_{\mathbf{z}}\,\mathbf{Z} \quad \longleftarrow & \operatorname{Ker} I''_o(\mathbf{z}) \;=\; T_{\mathbf{z}}\,\mathbf{Z} \qquad (\text{Lemma } 2.24\,.) \\
\downarrow & \\
\mathbf{W}_{\mathbf{z}} \;:=\; (\,T_{\mathbf{z}}\,\mathbf{Z})^{\perp} & ((2.25)\,; \;\; \textit{write} \;\; \mathcal{D}^{1,\,2} \;=\; T_{\mathbf{z}}\,\mathbf{Z} \oplus W_{\mathbf{z}}\,.) \\
\downarrow & \\
P_{\mathbf{z}} : \mathcal{D}^{1,\,2} \to \mathbf{W}_{\mathbf{z}} & (\textit{Projection unto the normal.}) \\
\downarrow & \\
P_{\mathbf{z}} \circ I'_\varepsilon(\mathbf{z} + w_\varepsilon(\mathbf{z})) \;=\; 0 & \leftarrow \;\; \textit{The auxiliary equation. Solution} : w_\varepsilon(\mathbf{z}) \in W_{\mathbf{z}}\,. \\
| & (\textit{Cancelation along the normal directions.}) \\
\downarrow & \\
\Phi_\varepsilon(\mathbf{z}) \;:=\; I_\varepsilon(\mathbf{z} + w_\varepsilon(\mathbf{z})) & (\textit{The finite dimensional reduction}\,; \;\; \mathbf{z} \in \mathbf{Z}\,.) \\
\downarrow & \\
\Phi'_\varepsilon(\mathbf{z}_\varepsilon + w_\varepsilon(\mathbf{z}_\varepsilon)) \;=\; 0 & (\textit{Critical point } \mathbf{z}_\varepsilon \textit{ of the finite dimensional functional.}) \\
\downarrow & \\
I'_\varepsilon(\mathbf{z}_\varepsilon + w_\varepsilon(\mathbf{z}_\varepsilon)) \;=\; 0 & (\,\textit{If such } \mathbf{z}_\varepsilon \textit{ exists for } |\,\varepsilon| \textit{ small enough}\,.) \\
\downarrow & \\
z_\varepsilon + w_\varepsilon(\mathbf{z}_\varepsilon) & \textit{is a solution of equation } (1.5) \textit{ with } (1.6)\,. \\
 & (\textit{Refer to } \text{Theorem } 2.27\,.)
\end{array}
$$

Theorem 2.27. *Keeping the notations in the chart, given a compact subset* $\mathbf{Z_c} \subset \mathbf{Z}$*, assume that* Φ_ε *has, for* $|\,\varepsilon|$ *sufficiently small, a critical point* $\mathbf{z}_\varepsilon \in \mathbf{Z}_c$*. Then* $\mathbf{z}_\varepsilon + w_\varepsilon(\mathbf{z}_\varepsilon)$ *is a critical point of* I_ε*.*

For the proof of Theorem 2.27, we refer to [4] (in particular, Theorem 2.12, Lemma 5.2 [pp. 61] and the discussion in pp. 59 – 60 , loc. cit.) .

§ 2 f. *Restriction of G to $\mathbf{Z}$.* With the structure

$$
\Phi_\varepsilon(\mathbf{z}) \;=\; I_o(\mathbf{z} + w_\varepsilon(\mathbf{z})) + \varepsilon\, G(\mathbf{z} + w_\varepsilon(\mathbf{z}))\,,
$$

the finite dimension reduction can be simplified further. Consider the restriction of G to $\mathbf{Z}$:

$$(2.28) \qquad G_{|\mathbf{Z}} \; : \; \mathbf{Z} \to \mathbb{R}$$

$$G_{|\mathbf{Z}}(\mathbf{z}) = G_{|\mathbf{Z}}(V_{\lambda,\,\xi}) = \bar{c}_{-1}\int_{R^n} H(y)\,[V_{\lambda,\xi}]^{\frac{2n}{n-2}} \qquad \text{(see (2.14))}$$

$$= \bar{c}_{-1}\int_{R^n} H(y)\left[\frac{\lambda}{\lambda^2 + |\,y-\xi|^2}\right]^n \quad \text{for } V_{\lambda,\,\xi} \in \mathbf{Z} \quad \text{(see (2.17))}.$$

It is shown in [4] that $G_{|\mathbf{Z}}$ and $G'_{|\mathbf{Z}}$ are good approximations of Φ_ε and Φ'_ε on $\mathbf{Z}$, respectively. Precisely, for $\mathbf{z} \in \mathbf{Z}$,

$$\Phi_\varepsilon(\mathbf{z}) = I_o(\mathbf{z}) + \varepsilon\, G_{|}(\mathbf{z}) + O(\varepsilon) \qquad [\text{refer to Lemma 2.15 in [4], pp. 24}]$$

$$\Phi'_\varepsilon(\mathbf{z}) = \varepsilon\, G'_{|\mathbf{Z}}(\mathbf{z}) + O(\varepsilon), \qquad (\text{the derivatives act on } T_{\mathbf{z}}\mathbf{Z};\ \text{refer to the proof of Theorem 2.17 in [4], pp. 26}).$$

Theorem 2.29. *Let* $G'_{|\mathbf{Z}}(\bar{\mathbf{z}}) = 0$. *Suppose one of the following conditions holds.*

(a) $\bar{\mathbf{z}}$ *is a strict local maximum or minimum of* $G_{|\mathbf{Z}}$.

(b) *There is an open and bounded set* $\mathcal{N} \subset \mathbf{Z}$ *such that* $\bar{\mathbf{z}} \in \mathcal{N}$ *and*

$$\deg\,(G'_{|\mathbf{Z}},\, \mathcal{N},\, \mathbf{0}) \neq 0.$$

Then for $|\,\varepsilon|$ *small enough,* Φ_ε *has a critical point.* (*It follows from Theorem* 2.27 *that* I_ε *also has a critical point.*)

See Theorem 2.16 and Theorem 2.17 in [4]. Via the parametrization (2.19), we identify $G_{|\mathbf{Z}}$ as a map from $\mathbb{R}^+ \times \mathbb{R}^n$ to $\mathbb{R}$. In particular, $\bar{\mathbf{z}} = V_{\bar\lambda,\,\bar\xi}$ is a critical point of $G_{|\mathbf{Z}}$ if and only if

$$(2.30) \qquad \frac{\partial G_{|\mathbf{Z}}}{\partial \lambda}(\bar\lambda,\,\bar\xi) = 0, \quad \text{and} \quad \frac{\partial G_{|\mathbf{Z}}}{\partial\, \xi_j}(\bar\lambda,\,\bar\xi) = 0 \quad \text{for } j = 1,\, 2,\, \cdots,\, n.$$

Refer to § A.4.2 in the e-appendix. With this, we can bring the discussion to $\mathbb{R}^+ \times \mathbb{R}^n$. In particular, we have the following.

Definition 2.31. *A critical point* $\mathbf{p} := (\bar\lambda,\, \bar\xi) \in \mathbb{R}^+ \times \mathbb{R}^n$ *of* $G_{|\mathbf{Z}}$ *is called* stable *if there exists a ball* $B_{\mathbf{p}}(\rho) \subset \mathbb{R}^+ \times \mathbb{R}^n$ *such that* $\deg\,(\bigtriangledown_{(\lambda,\,\xi)}\, G_{|\mathbf{Z}},\; B_{\mathbf{p}}(\rho),\; \vec{0}\,) \neq 0$.

Using the parametrization (2.19), one infers that a stable critical point of $G_{|\mathbf{Z}}$ fulfills condition **(b)** in Theorem 2.29. We refer to § A.3 in the e-Appendix for selected fundamental properties on the degree of a map. See also § A.4.2 in the e-appendix concerning the Hessian and the Jacobian.

§ 3. Tangent space, hyperbolic structure, and the normal space.

§ 3 a. *The tangent space.* For a point $\mathbf{z} = V_{\lambda,\xi} \in \mathbf{Z}$, via (2.19), the tangent space $T_{\mathbf{z}}\,\mathbf{Z}$ is spanned by the $(n+1)$ functions:

$$\text{(3.1)} \quad \varphi_o := \frac{\partial V_{\lambda,\xi}}{\partial \lambda} = \frac{\partial}{\partial \lambda}\left[\left(\frac{\lambda}{\lambda^2 + |\,y-\xi\,|^2}\right)^{\frac{n-2}{2}}\right] = -\frac{n-2}{2}\cdot \lambda^{\frac{n-4}{2}} \cdot \frac{(\,\lambda^2 - |\,y-\xi\,|^2)}{(\,\lambda^2 + |\,y-\xi\,|^2)^{\frac{n}{2}}}\,,$$

$$\varphi_j := \frac{\partial V_{\lambda,\xi}}{\partial\, \xi_j} = \frac{\partial}{\partial\, \xi_j}\left[\left(\frac{\lambda}{\lambda^2 + |\,y-\xi\,|^2}\right)^{\frac{n-2}{2}}\right] = -\frac{n-2}{2}\cdot \lambda^{\frac{n-2}{2}} \cdot \frac{2\,(\xi_j - y_j)}{(\,\lambda^2 + |\,y-\xi\,|^2)^{\frac{n}{2}}}$$

for $j = 1, 2, \cdots, n$. A direct calculation shows that the collection is orthogonal. That is

$$\langle\, \varphi_o\,,\ \varphi_j \rangle_{\bigtriangledown} = 0 = \langle\, \varphi_i\,,\ \varphi_j \rangle_{\bigtriangledown} \qquad \text{for } 1 \le i \ne j \le n\,.$$

Moreover,

$$\text{(3.2)} \qquad \|\,\varphi_o\|^2_{\bigtriangledown} = \int_{\mathbb{R}^n} \left\langle \bigtriangledown \frac{\partial V_{\lambda,\xi}}{\partial \lambda}\,,\ \bigtriangledown \frac{\partial V_{\lambda,\xi}}{\partial \lambda} \right\rangle = \frac{\bar c_1^{\,2}}{\lambda^2}\,,$$

$$\text{(3.3)} \qquad \|\,\varphi_j\|^2_{\bigtriangledown} = \int_{\mathbb{R}^n} \left\langle \bigtriangledown \frac{\partial V_{\lambda,\xi}}{\partial\, \xi_j}\,,\ \bigtriangledown \frac{\partial V_{\lambda,\xi}}{\partial\, \xi_j} \right\rangle = \frac{\bar c_2^{\,2}}{\lambda^2} \qquad \text{for} \quad 1 \le j \le n\,.$$

Here the positive constants $\bar c_1$ and $\bar c_2$ depend on the dimension n only.

§ 3 b. *Geometry of* $\mathbf{Z}$ *and hyperbolic space.* On $\mathbb{R}^+ \times \mathbb{R}^n$ with coordinates

$$(\,\lambda\,,\ \xi_1\,, \cdots,\ \xi_n)\,,$$

indexed zero $\uparrow$

using (2.19) to pull back the metric defined in (3.2) and (3.3), we obtain an induced Riemannian metric on $\mathbb{R}^+ \times \mathbb{R}^n$:

$$\text{(3.4)} \qquad \frac{c_\ell}{\lambda^2}\cdot \delta_{\ell\, k} \quad \text{with} \quad c_o = \bar c_o\,, \quad c_1 = c_2 = \cdots = c_n = \bar c_1 \qquad (\,\lambda > 0)\,,$$

where $0 \le \ell\,,\ k \le n$. The curvature tensor calculation shows that the induced metric has *constant* sectional curvature equal to $-\dfrac{1}{\bar c_o}$. See § A.4 in the e-Appendix. The standard hyperbolic metric on $\mathbb{R}^+ \times \mathbb{R}^n$ is

$$\frac{1}{t^2}\,(dt^2 \otimes dy^2) \qquad \text{at} \quad (t\,,\,y) \in \mathbb{R}^+ \times \mathbb{R}^n.$$

Thus we have the isometry:

$$\left(\mathbb{R}^+ \times \mathbb{R}^n\,, \quad \frac{c_\ell}{\lambda^2}\cdot \delta_{\ell\,k}\right) \longrightarrow \left(\mathbb{R}^+ \times \mathbb{R}^n\,, \quad \bar c_o \cdot \left[\frac{1}{t^2}\,(\,dt^2 \times dy^2\,)\right]\right)$$

$$(\,\lambda\,,\ \xi) \quad \mapsto \quad (t\,,\ y) = \left(\lambda\,,\ \frac{\bar c_1}{\bar c_o}\cdot \xi\right).$$

The (rescaled) hyperbolic structure provides a complete (non-compact) metric on $\mathbf{Z}$.

§ 3 c. *Orthonormal Basis.* Following (3.1), let

$$
\text{(3.5)} \qquad q_o \;:=\; \frac{\partial V_{\lambda,\,\xi}}{\partial \lambda} \Big/ \left\| \frac{\partial V_{\lambda,\,\xi}}{\partial \lambda} \right\|_{\bigtriangledown} \;=\; \bar C_o \, \lambda \cdot \lambda^{\frac{n-4}{2}} \cdot \frac{(\,\lambda^2 - |\,y - \xi|^2)}{(\,\lambda^2 + |\,y - \xi|^2)^{\frac{n}{2}}}\,,
$$

$$
q_j \;:=\; \frac{\partial V_{\lambda,\,\xi}}{\partial\, \xi_j} \Big/ \left\| \frac{\partial V_{\lambda,\,\xi}}{\partial\, \xi_j} \right\|_{\bigtriangledown} \;=\; \bar C_1 \, \lambda \cdot \lambda^{\frac{n-2}{2}} \cdot \frac{2\,(\,\xi_j - y_j)}{(\,\lambda^2 + |\,y - \xi|^2)^{\frac{n}{2}}}
$$

for $j \;=\; 1,\; 2, \cdots,\; n$. Here $\bar C_o$ and $\bar C_1$ are positive constants depending on n only. Thus $\{\, q_\ell \,\}_{0 \le \ell \le n}$ forms an orthonormal basis for $T_{\mathbf z}\mathbf Z$ with $\mathbf z = V_{\lambda,\,\xi}$. Using (2.11), we find

$$
\left\| \frac{\partial q_\ell}{\partial \lambda} \right\|^2_{\bigtriangledown} \;=\; \left\langle \bigtriangledown\, \frac{\partial q_\ell}{\partial \lambda}\,,\; \bigtriangledown \frac{\partial q_\ell}{\partial \lambda} \right\rangle_{\bigtriangledown} \;=\; \left\langle \Delta\, \frac{\partial q_\ell}{\partial \lambda}\,,\; \frac{\partial q_\ell}{\partial \lambda} \right\rangle_{\bigtriangledown} \;=\; \left\langle \frac{\partial(\Delta\, q_\ell)}{\partial \lambda}\,,\; \frac{\partial q_\ell}{\partial \lambda} \right\rangle_{\bigtriangledown}\,,
$$

leading to

$$
\text{(3.6)} \qquad \left\| D_o\, q_\ell \right\|^2_{\bigtriangledown} \;=\; \left\| \frac{\partial q_\ell}{\partial \lambda} \right\|^2 \;=\; \frac{\bar C_2}{\lambda^2}\,.
$$

Likewise,

$$
\text{(3.7)} \qquad \left\| D_j\, q_\ell \right\|^2_{\bigtriangledown} \;=\; \left\| \frac{\partial q_\ell}{\partial\, \xi_j} \right\|^2 \;=\; \frac{\bar C_3}{\lambda^2} \qquad \text{for } \; j \;=\; 1,\; 2, \cdots,\; n\,.
$$

In (3.6) and (3.7), $\bar C_2$ and $\bar C_3$ are positive constants depending on n only. Refer to § A.4.3 in the e-Appendix for more details.

§ 3 d. *The normal space* $\mathbf W_{\mathbf z} \;=\; (T_{\mathbf z}\, \mathbf Z)^\perp$. By definition,

$$
\begin{aligned}
\mathbf W_{\mathbf z} \;=\; (T_{\mathbf z}\, \mathbf Z)^\perp \;&=\; \left\{ w \in \mathcal D^{1,\,2} \;\middle|\; \int_{\mathbb R^n} \langle \bigtriangledown w\,,\, \bigtriangledown \varphi_\ell \rangle \;=\; 0 \quad \text{for } \; \ell \;=\; 0,\; 1, \cdots,\; n \right\} \\
&=\; \left\{ w \in \mathcal D^{1,\,2} \;\middle|\; \int_{\mathbb R^n} w\, \Delta\, \varphi_i \;=\; 0 \quad \text{for } \; \ell \;=\; 0,\; 1, \cdots,\; n \right\}.
\end{aligned}
$$

Here we apply (2.11) and use (3.1) to check condition (2.10). Define

$$
\text{(3.8)} \qquad \psi_o \;=\; \lambda^{\frac{n}{2}+1} \cdot \frac{(\,\lambda^2 - |\,y - \xi|^2)}{(\,\lambda^2 + |\,y - \xi|^2)^{\frac{n}{2}+2}}\,, \qquad \psi_j = \lambda^{\frac{n}{2}+2} \cdot \frac{2\,(y_i - \xi_i)}{(\,\lambda^2 + |\,y - \xi|^2)^{\frac{n}{2}+2}}
$$

for $j \;=\; 1,\; \cdots,\; n$. These are constants times $\Delta\, \varphi_\ell$. See § A.4.3 in the e-Appendix. The power on λ are chosen so that their $L^{\frac{2n}{n-2}}\,(\mathbb R^n)$ - norms are constant (independent on λ and ξ). We can express

$$
\text{(3.9)} \quad \mathbf W_{\mathbf z} \;=\; \left\{ w \in \mathcal D^{1,\,2} \;\middle|\; \int_{\mathbb R^n} w\, \psi_\ell \;=\; 0 \;\text{ for }\; \ell \;=\; 0,\; 1, \cdots,\; n \right\} \;\text{ for }\; \mathbf z \;=\; V_{\lambda,\,\xi} \in \mathbf Z\,.
$$

Let us continue with the observation:

$$\text{(3.10)}\quad \int_{\mathbb{R}^n} V_{\lambda,\,\xi}^{\frac{2n}{n-2}} \;=\; \text{Const.} \;\Longrightarrow\; D_\ell \left[\int_{\mathbb{R}^n} V_{\lambda,\,\xi}^{\frac{2n}{n-2}} \right] \;=\; 0 \qquad \text{for } \ell \;=\; 0\,,\; 1\,,\; \cdots,\; n$$

$$\Longrightarrow\; \int_{\mathbb{R}^n} V_{\lambda,\,\xi}^{\frac{n+2}{n-2}} \left[D_\ell\, V_{\lambda,\,\xi} \right] \;=\; 0 \quad \text{for } \ell \;=\; 0\,,\; 1\,,\; \cdots,\; n$$

$$\Longrightarrow\; \int_{\mathbb{R}^n} [\,\Delta_o\, V_{\lambda,\,\xi}]\,[D_\ell\, V_{\lambda,\,\xi}] \;=\; 0 \qquad [\,\text{equation } (1.4)\,]$$

$$\Longrightarrow\; \langle\, V_{\lambda,\,\xi}\,,\; \phi_\ell\,\rangle_{\bigtriangledown} \;=\; 0 \qquad [\,\text{via } (2.8)\ \&\ (2.11)\,]$$

$$\Longrightarrow\; \mathbf{z} \;=\; V_{\lambda,\,\xi} \in \mathbf{W}_{\mathbf{z}}\,.$$

This reminds us the sphere in $\mathbb{R}^n$, where any point on the sphere is consider as a vector which is perpendicular to the tangent space of the sphere at that point.

The following property concerning $\mathbf{W}_{\mathbf{z}}$ is a first step toward *uniform cancelation* in the Lyapunov - Schmidt reduction method for equation (1.1). Refer to [4], [8] and the references therein.

Proposition 3.11. *There exists a positive number* $\bar c_2$ *such that the inequality*

$$\text{(3.12)}\qquad \int_{\mathbb{R}^n} \Big(|\bigtriangledown\, w|^2 - n\,(n+2)\, V_{\lambda,\xi}^{\frac{4}{n-2}}\, w^2 \Big) \;\ge\; 2\,\bar c_2\, \|w\|_{\bigtriangledown}^2 \;-\; \frac{16 n^2}{\bar c_2} \Big(\int_{\mathbb{R}^n} V_{\lambda,\xi}^{\frac{n+2}{n-2}}\, w \Big)^2$$

(*for all* $w \in \mathbf{W}_{\mathbf{z}}$) *holds uniformly for all* $\mathbf{z} \;=\; V_{\lambda,\,\xi} \in \mathbf{Z}$ (*that is,* $\bar c_2$ *does not depend on* $\mathbf{z} \in \mathbf{Z}\,,$ *or* $w \in \mathbf{W}_{\mathbf{z}}$) .

Corollary 3.13. *For* $w \perp (\,\text{span}\,\{\mathbf{z}\} \oplus T_{\mathbf{z}}\, \mathbf{Z}\,)$ *we have*

$$\text{(3.14)}\qquad \int_{\mathbb{R}^n} \left[|\bigtriangledown\, w|^2 \;-\; n\,(n+2)\, V_{\lambda,\xi}^{\frac{4}{n-2}}\, w^2 \right] \;\ge\; 2\,\bar c_2\, \|\, w\|_{\bigtriangledown}^2\,,$$

uniformly for all $\mathbf{z} \;=\; V_{\lambda,\,\xi} \in \mathbf{Z}$. (*Here* $\bar c_2$ *is the same constant appeared in Proposition* 3.11.)

As the proof of the corollary is well - known among experts, we place it in the e-Appendix (§ A.4. d) for the interested readers to view. Finally, consider $w \;=\; \mathbf{z} = V_{\lambda,\,\xi}$ [recall that (3.10) shows that $\mathbf{z} \in \mathbf{W}_{\mathbf{z}}$.] Let us check what happens to the left hand side of (3.14) . To see precisely how (3.14) fails to hold when $w \;=\; \mathbf{z} \;=\; V_{\lambda\,,\,\xi}\,,$ we observe

$$V_{\lambda,\,\xi}\, \Delta\, V_{\lambda,\,\xi} \;+\; n\,(n-2)\, V_{\lambda,\xi}^{\frac{2n}{n-2}} \;=\; 0$$

$$\Longrightarrow\qquad \int_{\mathbb{R}^n} \left[|\bigtriangledown\, V_{\lambda,\xi}|^2 \;-\; n(n-2) V_{\lambda,\xi}^{\frac{4}{n-2}} \cdot V_{\lambda,\,\xi}^2 \right] \;=\; 0 \qquad (\text{integration by parts})$$

$$\Longrightarrow\qquad \int_{\mathbb{R}^n} \left[|\bigtriangledown\, V_{\lambda,\xi}|^2 \;-\; n\,(n+2) V_{\lambda,\xi}^{\frac{4}{n-2}} \cdot V_{\lambda,\,\xi}^2 \right] \;=\; -\,4n \int_{\mathbb{R}^n} V_{\lambda,\,\xi}^{\frac{2n}{n-2}}$$

$$=\; \frac{4n}{n(n-2)} \int_{\mathbb{R}^n} (\Delta\, V_{\lambda,\xi})\, V_{\lambda,\,\xi} \;=\; -\,\frac{4}{n-2} \int_{\mathbb{R}^n} |\bigtriangledown\, \mathbf{z}|^2 \;=\; -\,\frac{4}{n-2}\, \|\,\mathbf{z}\|_{\bigtriangledown}\,.$$

§ 4. Uniform cancelation.

§ 4 a. *Uniform solvability of the auxiliary equation.* The following result is from Lemma 2.21 in [4], pp. 27. See also (2.26) in § 2 d .

Lemma 4.1. *Refer to* (2.23), Lemma 2.24 *and* (2.26). *Assume conditions* (**i**) – (**iii**).

(**i**) $I_o''(\mathbf{z}) : \mathbf{W_z} \to \mathbf{W_z}$ *is uniformly invertible in* $\mathbf{z}$. *That is,*

$$\| \, (I_o''(\mathbf{z}))^{-1} \|_{L\,(\mathbf{W_z},\,\mathbf{W_z})} \ \le \ C \qquad \textit{for all} \ \ \mathbf{z} \in \mathbf{Z}\,. \tag{4.2}$$

(**ii**) *The remainder in*

$$I_o'(\mathbf{z} + w) - I_o''(\mathbf{z})\,[w] \qquad \textit{is uniformly small in} \ \ \mathbf{z}\,. \tag{4.3}$$

(**iii**) $\| \, P_{\mathbf{z}} \, G' \, (\mathbf{z} + w) \|_{\bigtriangledown} \ \le \ C$ *uniformly in* $\mathbf{z}$ (*for all* $\|w\|_{\bigtriangledown} \le 1$) .

Then there exists $\bar{\varepsilon}_1 > 0$ *such that for every* $|\,\varepsilon| \le \bar{\varepsilon}_1$, *and for every* $\mathbf{z} \in \mathbf{Z}$, *the auxiliary equation*

$$P_{\mathbf{z}} \circ I_\varepsilon' \, (\mathbf{z} + w) \ = \ 0 \tag{4.4}$$

has a unique solution $w_\varepsilon(\mathbf{z})$ *satisfying*

$$\|w_\varepsilon(\mathbf{z})\|_{\bigtriangledown} \to 0 \qquad \textit{uniformly as} \ \ |\,\varepsilon| \to 0\,. \tag{4.5}$$

In the next two sections (§ 4 b and § 4 c) , we verify conditions **(i)** – **(iii)** for the functional I_ε defined in (2.12) .

§ 4 b. *Uniform invertibility of* $I_o''(\mathbf{z})$*.* The following result is a direct consequence of Proposition 3.11 and Corollary 3.13. For the proof, see [4], or [8], or § A.5. a in the e-Appendix.

Proposition 4.6. *There exists a positive constant* $\bar{c}_3$ *such that for all* $\mathbf{z} \in \mathbf{Z}$,

$$\| \, I_o''(\mathbf{z})\,[f] \, \|_{\bigtriangledown} \ \ge \ \bar{c}_3 \, \|f\|_{\bigtriangledown} \qquad \textit{for all} \quad f \in \mathbf{W_z}\,. \tag{4.7}$$

§ 4 c. *Uniform approximation and bounded projection.* Likewise, the demonstrations of conditions (**ii**) and (**iii**) in Lemma 4.1 are rather standard. We prefer to put the details in § A.5 and § A.6 in the e-Appendix. Precisely, we show in the e-Appendix that

$$R_{\mathbf{z}}(w) \ := \ I_o'(\mathbf{z} + w) \ - \ I_o''(\mathbf{z})\,[w] \quad \text{satisfies} \quad \| \, R_{\mathbf{z}}(w) \| \ = \ o\,(\|w\|_{\bigtriangledown}) \tag{4.8}$$

as $\| w\|_{\bigtriangledown} \to 0$, uniformly for $\mathbf{z} \in \mathbf{Z}$. Moreover,

$$\| G'(\mathbf{z}+w)\| \le C \qquad \text{for all} \quad \mathbf{z} \in \mathbf{Z} \quad \text{and } w \text{ with } \| w\|_{\bigtriangledown} \le 1. \tag{4.9}$$

§ 4 d. *Uniform bounds in the expansions of I_o and G.* We also have the following uniform bounds.

$$I_o(\mathbf{z}+w_\varepsilon(\mathbf{z})) = \bar{c}_4 + o(\|w_\varepsilon(\mathbf{z})\|); \tag{4.10}$$

$$G(\mathbf{z}+w_\varepsilon(\mathbf{z})) = G(\mathbf{z}) + [\, G'(\mathbf{z}) \,|\, w_\varepsilon(\mathbf{z})\,] + o(\|w_\varepsilon(\mathbf{z})\|); \tag{4.11}$$

$$I_o'(\mathbf{z} + w_\varepsilon(\mathbf{z})) = [\, I_o''(\mathbf{z}) \,|\, w_\varepsilon(\mathbf{z})\,] + o(\|w_\varepsilon(\mathbf{z})\|); \tag{4.12}$$

$$G'(\mathbf{z} + w_\varepsilon(\mathbf{z})) = G'(\mathbf{z}) + [\, G''(\mathbf{z}) \,|\, w_\varepsilon(\mathbf{z})\,] + o(\|w_\varepsilon(\mathbf{z})\|). \tag{4.13}$$

Here $w_\varepsilon(\mathbf{z})$ is the solution to the auxiliary equation (4.4) described in Lemma 4.1, and

$$\bar{c}_4 = (n-2)\int_{\mathbb{R}^n} \left(\frac{1}{1+|\,y|^2} \right)^n . \tag{4.14}$$

We refer to § A .5 and § A .7 in the e-Appendix for more detail.

§ 4 e. *Uniform cancelation.*

Proposition 4.15. *There exists a positive constant $\bar{\varepsilon}_2$ such that for all ε with $|\,\varepsilon| \le \bar{\varepsilon}_2$, if Φ_ε has a sequence of critical points at $\mathbf{z}_{\varepsilon,i} \in \mathbf{Z}$, $i = 1, 2, \cdots$, then $\mathbf{z}_{\varepsilon,i} + w_\varepsilon(\mathbf{z}_{\varepsilon,i})$ are critical points of I_ε for all i. Here $w_\varepsilon(\mathbf{z}_{\varepsilon,i})$ is the solution of the auxiliary equation* (4.4) *as described in Lemma* 4.1.

See § A . 8 in the e-Appendix for the argument, which closely proceeds as in the proof of Theorem 2.12 in [4].

§ 4 f. *Uniform levelness.* We show that the solutions of the auxiliary equation (4.4) as described in Lemma 4.1 stay uniformly close to $\mathbf{Z}$, and " is almost parallel" to $\mathbf{Z}$. Precisely we have the following result.

Lemma 4. 16. *For $n \ge 6$, let $w_\varepsilon(\mathbf{z})$ be the unique solution of the auxiliary equation* (4.4) *as described in Lemma* 4.1. *There exists a positive number $\bar{\varepsilon}_3$ such that for every varepsilon satisfying $|\,\varepsilon| \le \bar{\varepsilon}_3$, we have*

$$\| w_\varepsilon(\mathbf{z})\|_{\bigtriangledown} \le \bar{C}_4 \,|\,\varepsilon| \quad \textit{and} \quad \| D_\ell\, w_\varepsilon(\mathbf{z})\|_{\bigtriangledown} \le \frac{\bar{C}_4}{\lambda}\,|\,\varepsilon|^{\frac{4}{n-2}} \quad \textit{for } \ell = 0, 1, \cdots, n. \tag{4.17}$$

Here $\bar{C}_4$ is a positive constant independent on $\mathbf{z} = V_{\lambda,\,\xi} \in \mathbf{Z}$.

Proof. We argue as in the proof of Lemma 2.11 in [4], pp. 21. Let us begin with

$$I_\varepsilon'(\mathbf{z}+w_\varepsilon(\mathbf{z})) \qquad [\, = I_o'(\mathbf{z}+w_\varepsilon(\mathbf{z})) + \varepsilon \cdot G'(\mathbf{z}+w_\varepsilon(\mathbf{z}))\,] \tag{4.18}$$

$$= \; I'_o(\mathbf{z}) \; + \; I''_o(\mathbf{z})\,[\,w_\varepsilon(\mathbf{z})\,] + \; \varepsilon\,\{\,G'(\mathbf{z}) + \; G''(\mathbf{z})[\,w_\varepsilon(\mathbf{z})\,]\,\} \; + \; o\,(\|\,w_\varepsilon(\mathbf{z})\|_{\,\triangledown})$$

$$[\,\text{via (4.12) \& (4.13)}\,]$$

$$= \; \varepsilon\,G'(\mathbf{z}) \; + \; I''_o(\mathbf{z})\,[w_\varepsilon(\mathbf{z})\,] \; + \; \varepsilon\,G''(\mathbf{z})\,[w_\varepsilon(\mathbf{z})\,] \; + \; o\,(\|\,w_\varepsilon(\mathbf{z})\|_{\,\triangledown}) \quad \text{uniformly} \quad \text{in} \;\; \mathbf{z}\,.$$

From the auxiliary equation (4.4), we have the following.

$$(4.19) \qquad P_{\mathbf{z}} \circ I'_\varepsilon(\mathbf{z} + w_\varepsilon(\mathbf{z})) \; = \; 0$$

$$\Longrightarrow \; \varepsilon\,P_{\mathbf{z}} \circ G'(\mathbf{z}) \; + \; P_{\mathbf{z}} \circ I''_o(\mathbf{z})\,[\,w_\varepsilon(\mathbf{z})\,] + \varepsilon\,P_{\mathbf{z}} \circ G''(\mathbf{z})\,[w_\varepsilon(\mathbf{z})\,] \; + \; o\,(\|\,w_\varepsilon(\mathbf{z})\|_{\,\triangledown}) \; = \; 0$$

$$\Longrightarrow \; P_{\mathbf{z}} \circ G'(\mathbf{z}) + \; P_{\mathbf{z}} \circ I''_o(\mathbf{z})\,[\,\varepsilon^{-1}\,w_\varepsilon(\mathbf{z})\,] + \; P_{\mathbf{z}} \circ G''(\mathbf{z})[\,w_\varepsilon(\mathbf{z})\,] + \; o\,(\|\,\varepsilon^{-1}w_\varepsilon(\mathbf{z})\|_{\,\triangledown}) \; = \; 0.$$

Via (4.5), we already know that $\|\,w_\varepsilon(\mathbf{z})\|_{\,\triangledown}$ is uniformly small, hence

$$(4.20) \qquad P_{\mathbf{z}} \circ I''_o(\mathbf{z})\,[\,\varepsilon^{-1}w_\varepsilon(\mathbf{z})\,] \; = \; P_{\mathbf{z}} \circ G'(\mathbf{z}) \; + \; o\,(1) \; + \; o\,(\|\,\varepsilon^{-1}w_\varepsilon(\mathbf{z})\|_{\,\triangledown})\,.$$

Compare with (4.19), the small order terms in (4.20) are in $\mathbf{W}_{\mathbf{z}}$. The operator is identified via Riesz Representation Theorem with a vector in $\mathcal{D}^{1,\,2}$. We check in § A.9 in the e-Appendix that $G'(\mathbf{z})$ (as an operator) is uniformly bounded. Since

$$P_{\mathbf{z}} \circ I''_o(\mathbf{z}) \; : \mathbf{W}_{\mathbf{z}} \to \mathbf{W}_{\mathbf{z}}$$

is uniformly invertible [refer to Proposition 4.6, (2.23) and (2.26)], it follows that

$$\|\,\varepsilon^{-1} \cdot w_\varepsilon(\mathbf{z})\|_{\,\triangledown}$$

is uniformly bounded.

As for the second estimates in (4.17), Lemma 2.11 in [4] asserts that $w_\varepsilon(\mathbf{z})$ is C^1 in $\mathbf{z}$. Via the parametrization (2.19) it is differentiable (C^1) with respective to λ and ξ. We make use of the inequality:

$$(4.21) \qquad n \; \ge \; 6 \quad \Longleftrightarrow \quad \frac{4}{n-2} \; \le \; 1 \; \Longrightarrow \; \left|\,|\,a + b\,|^{\frac{4}{n-2}} - a^{\frac{4}{n-2}}\right| \; \le \; |\,b|^{\frac{4}{n-2}}$$

for $a > 0$ and $b \in \mathbb{R}$. Refer to Lemma 6.18 in [4], pp. 93 (for the definition of 2^* used in the book, see (6.9), pp. 75, loc. cit.).

Differentiating the auxiliary equation (4.4), as $P_{\mathbf{z}}$ is linear, we obtain

$$(4.22) \;\; P_{\mathbf{z}} \circ I''_o(\mathbf{z} + w_\varepsilon(\mathbf{z}))\,[\,D_\ell\,\mathbf{z} + D_\ell\,w_\varepsilon(\mathbf{z})\,] \; + \; \varepsilon\,P_{\mathbf{z}} \circ G''(\mathbf{z} + w_\varepsilon(\mathbf{z}))\,[\,D_\ell\;\mathbf{z} + D_\ell\,w_\varepsilon(\mathbf{z})\,] \; = \; 0$$

for $\ell \; = \; 0\,,\; 1,\; \cdots,\; n$. From (2.20) we recognize

$$(4.23) \quad (I''_o(\mathbf{z} + w_\varepsilon(\mathbf{z}))\,[D_\ell\,\mathbf{z} + D_\ell\,w_\varepsilon(\mathbf{z})\,]\;h) \qquad\qquad (h \, \in \, \mathcal{D}^{1,\,2})$$

$$= \int_{\mathbb{R}^n} \left[\,\langle \triangledown\,[D_\ell\,\mathbf{z} + D_\ell\,w_\varepsilon(\mathbf{z})\,]\,,\, \triangledown\,h\rangle \; - \; n\,(n+2)\,[\,\mathbf{z} + w_\varepsilon(\mathbf{z})\,]_+^{\frac{4}{n-2}}\;[\,D_\ell\,\mathbf{z} + D_\ell\,w_\varepsilon(\mathbf{z})\,] \cdot h\,\right]$$

$$= \int_{{\rm I\!R}^n} \langle \bigtriangledown [D_\ell \, w_\varepsilon(\mathbf{z})\,]\,,\, \bigtriangledown h \rangle - n\,(n+2) \int_{{\rm I\!R}^n} \mathbf{z}^{\frac{4}{n-2}}\,[\,D_\ell \, w_\varepsilon(\mathbf{z})\,] \cdot h$$

$$+ \int_{{\rm I\!R}^n} \langle \bigtriangledown [D_\ell \, \mathbf{z}]\,,\, \bigtriangledown h \rangle - n\,(n+2) \int_{{\rm I\!R}^n} \mathbf{z}^{\frac{4}{n-2}}\,[\,D_\ell \, \mathbf{z}] \cdot h$$

$$- n\,(n+2) \int_{{\rm I\!R}^n} \left[(\mathbf{z} + w)_+^{\frac{4}{n-2}} - \mathbf{z}^{\frac{4}{n-2}} \right] [\,D_\ell \, \mathbf{z} + D_\ell \, w_\varepsilon(\mathbf{z})\,] \cdot h$$

$$= \ (I''_o(\mathbf{z})\,[D_\ell \, w_\varepsilon(\mathbf{z})\,]\ h) \ + \ (I''_o(\mathbf{z})\,[\,D_\ell \, z]\ h)$$

$$(\uparrow \ = \ 0 \ \text{ via } \ \text{Lemma } \ 2.24)$$

$$- n\,(n+2) \int_{{\rm I\!R}^n} \left[(\mathbf{z} + w_\varepsilon(\mathbf{z}))_+^{\frac{4}{n-2}} - \mathbf{z}^{\frac{4}{n-2}} \right] [\,D_\ell \ \mathbf{z} + D_\ell \, w_\varepsilon(\mathbf{z})\,] \cdot h\,.$$

Consider the last term in (4.23):

$$(4.24) \qquad \left| \int_{{\rm I\!R}^n} \left[(\mathbf{z} + w_\varepsilon(\mathbf{z}))_+^{\frac{4}{n-2}} \ - \ \mathbf{z}^{\frac{4}{n-2}} \right] [\,D_\ell \, \mathbf{z} \ + \ D_\ell \, w_\varepsilon(\mathbf{z})\,] \cdot h \right|$$

$$\leq \ \int_{{\rm I\!R}^n} \left| (\mathbf{z} + w_\varepsilon(\mathbf{z}))_+^{\frac{4}{n-2}} \ - \ \mathbf{z}^{\frac{4}{n-2}} \right| (\,|\,D_\ell \, \mathbf{z}| \ + \ |\,D_\ell \, w_\varepsilon(\mathbf{z})|\,) \cdot |\,h|$$

$$\Big\{ \text{if at a point } \ [\,\mathbf{z} + w_\varepsilon(\mathbf{z})\,]_+ \ = \ 0 \ \Longrightarrow \ |\,w_\varepsilon(\mathbf{z})| \ \geq \ \mathbf{z} = \ V_{\lambda,\,\xi} > 0\,;$$

$$\text{whereas at a point } \ [\,\mathbf{z} + w_\varepsilon(\mathbf{z})\,]_+ \ > \ 0 \Longrightarrow \ (\mathbf{z} + w_\varepsilon(\mathbf{z}))_+ \ = \ |\,\mathbf{z} \ + \ w_\varepsilon(\mathbf{z})| \Big\}$$

$$\leq \ C_1 \int_{{\rm I\!R}^n} |\,w_\varepsilon(\mathbf{z})|^{\frac{4}{n-2}}\,[\,|\,D_\ell \, \mathbf{z}| \ + \ |D_\ell \, w_\varepsilon(\mathbf{z})|\,] \cdot |\,h| \qquad [\,\text{using } \ (4.21)\,]$$

$$\leq \ C_2 \, \|\,h\|_{\bigtriangledown} \cdot (\|D_\ell \, \mathbf{z}\|_{\bigtriangledown} \ + \ \|D_\ell \, w_\varepsilon(\mathbf{z})\|_{\bigtriangledown}) \left(\int_{{\rm I\!R}^n} |\,w_\varepsilon(\mathbf{z})|^{\frac{2n}{n-2}} \right)^{\frac{2}{n}}$$

(apply Hölder's inequality twice; cf. § A.5 in the e-Appendix)

$$\leq \ C_3 \, \|\,h\|_{\bigtriangledown} \cdot (\|D_\ell \, \mathbf{z}\|_{\bigtriangledown} + \|D_\ell \, w_\varepsilon(\mathbf{z})\|_{\bigtriangledown}) \cdot \|w_\varepsilon(\mathbf{z})\|_{\bigtriangledown}^{\frac{4}{n-2}} \qquad \left[\frac{2}{n} \ = \left(\frac{n-2}{2n} \right) \cdot \frac{4}{n-2} \right]$$

$$\leq \ C_4 \, \|\,h\|_{\bigtriangledown} \cdot \left[\frac{1}{\lambda} + \|\,D_\ell \, w_\varepsilon(\mathbf{z})\|_{\bigtriangledown} \right] \cdot |\,\varepsilon|^{\frac{4}{n-2}} \qquad (\text{as ususal } \ \mathbf{z} \ = \ V_{\lambda,\,\xi}\,)\,.$$

In the above, we also apply (3.1) – (3.3), and the first estimate in (4.17).

For the second term in the right hand side of (4.22), from (2.14), we have

$$(\,G''(f)\,[\,h]\,\phi) \ = \ - \, \tilde{c}_n \cdot \frac{n+2}{n-2} \cdot \int_{{\rm I\!R}^n} H \cdot f_+^{\frac{4}{n-2}} \, g \cdot h \qquad \text{for } \ f\,,\ h\,,\ \phi \in \mathcal{D}^{1,\,2}$$

$$\Longrightarrow \qquad (\,G''\,(\mathbf{z} + w_\varepsilon(\mathbf{z}))\,[\,D_\ell \, \mathbf{z} + D_\ell \, w_\varepsilon(\mathbf{z})\,]\ h)$$

$$= \ \frac{n+2}{n-2} \, \tilde{c}_n \int_{{\rm I\!R}^n} H \cdot [(\mathbf{z} + w_\varepsilon(\mathbf{z})\,]_+^{\frac{4}{n-2}} \ (D_\ell \, \mathbf{z} + D_\ell \, w_\varepsilon(\mathbf{z})) \cdot h\,.$$

Let us continue as in (4.24):

$$(4.25) \qquad \left| \int_{{\rm I\!R}^n} H \cdot [\,(\mathbf{z} + w_\varepsilon(\mathbf{z})\,]_+^{\frac{4}{n-2}} \ (D_\ell \, \mathbf{z} + D_\ell \, w_\varepsilon(\mathbf{z})) \cdot h \right|$$

$$\leq \; C_1 \int_{\mathbb{R}^n} \Big[\mathbf{z}^{\frac{4}{n-2}} + [\, w_\varepsilon(\mathbf{z})\,]^{\frac{4}{n-2}} \Big] \cdot (|D_\ell\,\mathbf{z}| + |D_\ell\, w_\varepsilon(\mathbf{z})|\,) \cdot |\,h|$$

$$\leq \; C_2 \left[\frac{1}{\lambda} + \|D_\ell\, w_\varepsilon(\mathbf{z})\|_{\bigtriangledown} + |\,\varepsilon|^{\frac{4}{n-2}} \left(\frac{1}{\lambda} + \|D_\ell\, w_\varepsilon(\mathbf{z})\|_{\bigtriangledown} \right) \right] \cdot \|\,h\|_{\bigtriangledown}\,.$$

Combining the estimates (4.24) and (4.25), we obtain

$$P_{\mathbf{z}} \circ I''_o\,(\mathbf{z})[D_\ell\, w_\varepsilon(\mathbf{z})\,] \;=\; O\left(\frac{|\,\varepsilon|^{\frac{4}{n-2}}}{\lambda} \right) + O\left(|\,\varepsilon|^{\frac{4}{n-2}} \cdot \|D_\ell\, w_\varepsilon(\mathbf{z})\|_{\bigtriangledown} \right) \qquad \text{for} \;\; n \geq 6\,.$$

Write

$$D_\ell\, w_\varepsilon(\mathbf{z}) \;=\; [D_\ell\, w_\varepsilon(\mathbf{z})\,]_\perp + [D_\ell\, w_\varepsilon(\mathbf{z})\,]_{\,T_{\mathbf{z}}(\mathbf{Z})}\,.$$

It follows from (2.21) and (2.23) that

$$\|\, I''_o\,(\mathbf{z})\,([\,D_\ell\, w_\varepsilon(\mathbf{z})\,]_{\,T_{\mathbf{z}}(\mathbf{Z})})\|_{\bigtriangledown} \leq C\|\, D_\ell\, w_\varepsilon(\mathbf{z})\,]_{\,T_{\mathbf{z}}(\mathbf{Z})}\|_{\bigtriangledown} \qquad \text{(uniformly in } \mathbf{z})\,.$$

From (4.5) in Lemma 4.1, we obtain [see also (A.8.1) in the e-Appendix],

$$\|\,[\,D_\ell\, w_\varepsilon(\mathbf{z})\,]_{\,T_{\mathbf{z}}(\mathbf{Z})}\|_{\bigtriangledown} \;\leq\; C_1 \cdot \frac{|\,\varepsilon|}{\lambda} \qquad \text{(uniformly in } \mathbf{z})$$

$$\Longrightarrow \quad \|\, I''_o\,(\mathbf{z})\,([\,D_\ell\, w_\varepsilon(\mathbf{z})\,]_{\,T_{\mathbf{z}}(\mathbf{Z})})\|_{\bigtriangledown} \;\leq\; C_2 \cdot \frac{|\,\varepsilon|}{\lambda} \qquad \text{(uniformly in } \mathbf{z})\,.$$

Hence

$$P_{\mathbf{z}} \,\circ\, I''_o\,(\mathbf{z})\,[\,D_\ell\, w_\varepsilon(\mathbf{z})\,]_\perp \;=\; O\left(\frac{|\,\varepsilon|^{\frac{4}{n-2}}}{\lambda} \right) + O\left(|\,\varepsilon|^{\frac{4}{n-2}} \cdot \|\,[\,D_\ell\, w_\varepsilon(\mathbf{z})\,]_\perp\|_{\bigtriangledown} \right)$$

uniformly in $\mathbf{z}$. [$I''_o\,(\mathbf{z})$, considered as a linear map from $\mathcal{D}^{1,\,2}$ to itself, is uniformly bounded in $\mathbf{z}$.] We obtain the desired estimate on $\|\, I''_o\,(\mathbf{z})\,(\,[\,D_\ell\, w_\varepsilon(\mathbf{z})\,]_\perp\,)\|_{\bigtriangledown}$ by using (2.26) and the uniform invertibility (4.7). □

§ 4 g. *Uniform existence for stable critical points.* The key merit of the following result is that we can allow $\bar\lambda \to 0^+$.

Theorem 4.26. *For $n \geq 6$, there exists positive constants $\bar C_5$ and $\bar\varepsilon_4$ such that the following holds. Suppose that $\bar{\mathbf{z}} \;=\; V_{\bar\lambda,\,\bar\xi} \in \mathbf{Z}$ is a stable critical point for $G_{|\mathbf{Z}}$, and there exists $\bar\rho > 0$ such that*

$$\text{(4.27)} \qquad B_{\bar\lambda,\,\bar\xi}\,(\bar\rho) \subset \mathbb{R}^+ \times \mathbb{R}^n \quad \textit{and} \quad \min_{(\lambda,\,\xi)\,\in\,\partial B_{\bar\lambda,\,\bar\xi}\,(\bar\rho)} |\bigtriangledown\, G_{|\mathbf{Z}}(\,\lambda,\,\xi)| > \frac{\bar C_5}{\lambda}\,.$$

[Here $\partial B_{\bar\lambda,\bar\xi}\,(\bar\rho)$ is the boundary of the ball.] Then for $|\,\varepsilon| < \bar\varepsilon_4$, the functional $\mathbf{I}_\varepsilon$ has a critical point at $\tilde{\mathbf{z}} + w_\varepsilon\,(\tilde{\mathbf{z}})$ with $\tilde{\mathbf{z}} \;=\; V_{\tilde\lambda,\,\tilde\xi}$, where $(\tilde\lambda,\;\tilde\xi) \in B_{\bar\lambda,\,\bar\xi}\,(\bar\rho)$. Here $w_\varepsilon\,(\tilde{\mathbf{z}})$ is

the solution of the auxiliary equation (4.4) *as described in Lemma* 4.1.

Proof. We first verify the claim that, for all $\mathbf{z} \in \mathbf{Z}$, we have

$$D_\ell \, \Phi_\varepsilon(\mathbf{z}) \; = \; \varepsilon \cdot [\, D_\ell \, G(\mathbf{z}) \,] \; + \; o\left(\frac{\varepsilon}{\lambda}\right) \qquad \text{uniformly} \quad \text{as} \quad \varepsilon \to 0\,. \tag{4.28}$$

Here $\ell \; = \; 0, \; 1, \cdots, \; n$. We start with the definition in (2.12):

$$\begin{aligned} \Phi_\varepsilon(\mathbf{z}) \; &= \; I_\varepsilon\,(\mathbf{z} + w_\varepsilon(\mathbf{z})) \\ = \quad & \int_{\mathbb{R}^n} \left\{ \frac{1}{2} \, \langle \, \nabla_o \; (\mathbf{z} + w_\varepsilon(\mathbf{z}))\,, \, \nabla_o \, (\mathbf{z} + w_\varepsilon(\mathbf{z}))\, \rangle \; - \; \frac{n-2}{2n} \cdot n\,(n-2)\,[\, z + w_\varepsilon(\mathbf{z})\,]_+^{\frac{2n}{n-2}} \right\} \\ & \qquad + \varepsilon \cdot \bar{c}_{-1} \int_{\mathbb{R}^n} H\,[\, \mathbf{z} + w_\varepsilon(\mathbf{z})\,]_+^{\frac{2n}{n-2}} \end{aligned}$$

$$\begin{aligned} \Longrightarrow \qquad D_\ell \, \Phi_\varepsilon(\mathbf{z}) = & \int_{\mathbb{R}^n} \Big\{ \langle \, \nabla_o \, [\, \mathbf{z} \, + \, w_\varepsilon(\mathbf{z})\,]\,, \;\; \nabla_o \, (D_\ell \, \mathbf{z} \; + \; D_\ell \, w_\varepsilon(\mathbf{z}))\, \rangle \\ & \qquad - \, n(n-2)\,(D_\ell \, \mathbf{z} \, + \, D_\ell \, w_\varepsilon(\mathbf{z})) \cdot [\, \mathbf{z} \, + \, w_\varepsilon(\mathbf{z})\,]_+^{\frac{n+2}{n-2}} \Big\} \\ & \qquad - \varepsilon \cdot \tilde{c}_n \int_{\mathbb{R}^n} H \cdot (D_\ell \, \mathbf{z} \, + \, D_\ell \, w_\varepsilon(\mathbf{z})) \cdot [\, \mathbf{z} \, + \, w_\varepsilon(\mathbf{z})\,]_+^{\frac{n+2}{n-2}} \end{aligned}$$

$$\begin{aligned} \Longrightarrow \qquad D_\ell \, \Phi_\varepsilon(\mathbf{z}) = & \int_{\mathbb{R}^n} \Big\{ \langle \nabla_o \, \mathbf{z} \,, \, \nabla_o \, D_\ell \, \mathbf{z} \rangle \, - \, n\,(n-2)(D_\ell \, \mathbf{z}) \, \mathbf{z}^{\frac{n+2}{n-2}} \Big\} \quad (= \; 0) \\ & \qquad\qquad [\, \text{from} \quad (1.3) \quad \& \quad (1.4) \quad \uparrow \,] \\ & + \int_{\mathbb{R}^n} \langle \, \nabla_o \; \mathbf{z} \,, \, \nabla_o \, D_\ell \, w_\varepsilon(\mathbf{z}) \rangle \; + \; \int_{\mathbb{R}^n} \langle \, \nabla_o \; w_\varepsilon(\mathbf{z}) \,, \, \nabla_o \, D_\ell \, \mathbf{z} \rangle \\ & \quad + \int_{\mathbb{R}^n} \langle \, \nabla_o \; w_\varepsilon(\mathbf{z}) \,, \, \nabla_o \, D_\ell \, w_\varepsilon(\mathbf{z}) \rangle \end{aligned}$$

$$\begin{aligned} & - n\,(n-2) \int_{\mathbb{R}^n} D_\ell \, \mathbf{z} \left[(\mathbf{z} \, + \, w_\varepsilon(\mathbf{z}))_+^{\frac{n+2}{n-2}} - \mathbf{z}^{\frac{n+2}{n-2}} \right] - \, n\,(n-2) \int_{\mathbb{R}^n} D_\ell \, w_\varepsilon(\mathbf{z}) \, [\, \mathbf{z} + w_\varepsilon(\mathbf{z})\,]_+^{\frac{n+2}{n-2}} \\ & \quad - \varepsilon \cdot \tilde{c}_n \int_{\mathbb{R}^n} H \, [\, D_\ell \, \mathbf{z}] \cdot \mathbf{z}^{\frac{n+2}{n-2}} \; - \; \varepsilon \cdot \tilde{c}_n \int_{\mathbb{R}^n} H \, [D_\ell \, \mathbf{z}] \cdot \left[(\mathbf{z} + w_\varepsilon(\mathbf{z}))_+^{\frac{n+2}{n-2}} - \mathbf{z}^{\frac{n+2}{n-2}} \right] \\ & \quad\quad - \varepsilon \cdot \tilde{c}_n \int_{\mathbb{R}^n} H \cdot D_\ell \, w_\varepsilon(\mathbf{z}) [\, \mathbf{z} + w_\varepsilon(\mathbf{z})\,]_+^{\frac{n+2}{n-2}} \end{aligned}$$

for $\ell \; = \; 0, \; 1, \cdots, \; n$. Together with equation (1.4) and expressions (3.1) – (3.3), we have

(4.29)

$$\begin{aligned} D_\ell \, \Phi_\varepsilon(\mathbf{z}) \quad = \quad & \varepsilon D_\ell \, G_{|\mathbf{z}}(\mathbf{z}) \, + \, \int_{\mathbb{R}^n} \langle \nabla_o \, w_\varepsilon(\mathbf{z}) \,, \, \nabla_o \, D_\ell \, \mathbf{z} \, \rangle - n\,(n+2) \cdot \mathbf{z}^{\frac{4}{n-2}} \, [\, D_\ell \, \mathbf{z}] \cdot w_\varepsilon(\mathbf{z}) \\ & \qquad [\uparrow \, = \, I_o''\,(\mathbf{z})\,[D_\ell \, \mathbf{z}\,] \, w_\varepsilon(\mathbf{z}) = 0 \text{ (Lemma 2.24)}\,] \\ & - n(n-2) \int_{\mathbb{R}^n} D_\ell \, \mathbf{z} \left[(\mathbf{z} + w_\varepsilon(\mathbf{z}))_+^{\frac{n+2}{n-2}} - \mathbf{z}^{\frac{n+2}{n-2}} - \frac{n+2}{n-2} \, \mathbf{z}^{\frac{4}{n-2}} \, w_\varepsilon(\mathbf{z}) \right] \end{aligned}$$

$$
\begin{aligned}
& - n(n-2) \int_{\mathbb{R}^n} D_\ell\, w_\varepsilon(\mathbf{z})\, (\mathbf{z} + w_\varepsilon(\mathbf{z}))_+^{\frac{n+2}{n-2}} + \int_{\mathbb{R}^n} \langle\, \bigtriangledown_o \mathbf{z}\,,\, \bigtriangledown_o D_\ell\, w_\varepsilon(\mathbf{z})\,\rangle \\
& \quad + \int_{\mathbb{R}^n} \langle \bigtriangledown_o w_\varepsilon(\mathbf{z})\,,\, \bigtriangledown_o D_\ell\, w_\varepsilon(\mathbf{z})\rangle \\
& \qquad - \varepsilon \cdot \tilde{c}_n \int_{\mathbb{R}^n} H\,[D_\ell\, \mathbf{z}] \cdot \left[(\mathbf{z} + w_\varepsilon(\mathbf{z}))_+^{\frac{n+2}{n-2}} - \mathbf{z}^{\frac{n+2}{n-2}} \right] \\
& \qquad\quad - \varepsilon \cdot \tilde{c}_n \int_{\mathbb{R}^n} H\, D_\ell\, w_\varepsilon(\mathbf{z}) [\, \mathbf{z} + w_\varepsilon(\mathbf{z})\,]_+^{\frac{n+2}{n-2}} \\
= \;\; & \varepsilon D_\ell\, G_{|\mathbf{z}}(\mathbf{z}) \; + \; \mathbf{I} \; + \; \mathbf{II} \; + \; \mathbf{III} \; + \; \mathbf{IV}\,,
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbf{I} \;\; &= \;\; -\,n\,(n-2) \int_{\mathbb{R}^n} D_\ell\; \mathbf{z}\, \left[(\mathbf{z} + w_\varepsilon(\mathbf{z}))_+^{\frac{n+2}{n-2}} - \mathbf{z}^{\frac{n+2}{n-2}} - \frac{n+2}{n-2}\, \mathbf{z}^{\frac{4}{n-2}}\, w_\varepsilon(\mathbf{z}) \right]\,, \\
\mathbf{II} \;\; &= \;\; -\,n\,(n-2) \int_{\mathbb{R}^n} D_\ell\, w_\varepsilon(\mathbf{z})\, \left[(\mathbf{z} + w_\varepsilon(\mathbf{z}))_+^{\frac{n+2}{n-2}} - \mathbf{z}^{\frac{n+2}{n-2}} - \frac{n+2}{n-2} \mathbf{z}^{\frac{4}{n-2}} w_\varepsilon(\mathbf{z}) \right]\,, \\
\mathbf{III} \;\; &= \;\; (I_o''\,(\mathbf{z})\,[w_\varepsilon(\mathbf{z})\,]\,(D_\ell\, w_\varepsilon(\mathbf{z})) - \; \varepsilon \cdot \tilde{c}_n \int_{\mathbb{R}^n} H\,[D_\ell\, \mathbf{z}] \cdot \left[(\mathbf{z} + w_\varepsilon(\mathbf{z}))_+^{\frac{n+2}{n-2}} - \mathbf{z}^{\frac{n+2}{n-2}} \right]\,, \\
\mathbf{IV} \;\; &= \;\; -\; \varepsilon \cdot \tilde{c}_n \int_{\mathbb{R}^n} H\, D_\ell\, w_\varepsilon(\mathbf{z}) [\, \mathbf{z} + w_\varepsilon(\mathbf{z})\,]_+^{\frac{n+2}{n-2}}\;.
\end{aligned}
$$

Here we use (1.4) and (2.21). For the terms marked (**I**) and (**II**) in (4.29), see (A.5.5) in § A.5 in the e-Appendix. The first term in (**III**) can be estimated by using the uniform bound on $\|\, I_o''(\mathbf{z})\|$ and the bounds in Lemma 4.16. Whereas the second term in (**III**) is handled as in (4.24), and (**IV**) as in (A.6.2) in the e-Appendix. Together with (3.2) and (3.3), we obtain

$$
D_\ell\, \Phi_\varepsilon(\mathbf{z}) \;=\; \varepsilon D_\ell\, G_{|\mathbf{z}}(\mathbf{z}) \,+\, O\left(|\,\varepsilon| \cdot \frac{|\,\varepsilon|}{\lambda} \right) \,+\, O\left(|\,\varepsilon| \cdot \frac{|\,\varepsilon|^{\frac{4}{n-2}}}{\lambda} \right) \qquad \text{for} \quad \ell \;=\; 0,\; 1, \cdots,\; n\,.
$$

Hence we establish the claim. Thus we can find $\bar{\varepsilon}_4$ small so that for $|\,\varepsilon| < \bar{\varepsilon}_4$, we have

$$
\|\, \bigtriangledown_{(\,\lambda,\,\xi)}\, \Phi_\varepsilon(\mathbf{z}) \;-\; \varepsilon \bigtriangledown_{(\,\lambda,\,\xi)}\, G_{|\mathbf{z}}(\mathbf{z})\| \;\le\; c_1 \cdot \frac{|\,\varepsilon|}{\lambda} \;<\; \bar{C}_5 \cdot \frac{|\,\varepsilon|}{\lambda} \qquad \text{for} \;\; (\,\lambda,\; \xi) \,\in\, \partial B_{\bar{\lambda},\; \bar{\xi}}\,(\bar{\rho})\,.
$$

Here, as usual, $\mathbf{z} \;=\; V_{\lambda,\; \xi}$. Applying the degree theory [14] (also described in § A.3.3 and § A.3.4 [in particular, (A.3.8)]), we obtain a critical point $\tilde{\mathbf{z}} \in B_{\bar{\lambda},\; \bar{\xi}}\,(\bar{\rho})$ for $\Phi_\varepsilon(\mathbf{z})$. Applying Proposition 4.15, we complete the proof of the theorem. □

§ 5. Separation and blow - up.

In this section, we provide criteria for a sequence of critical points $\{\mathbf{z}_i \,+\, w_\varepsilon(\mathbf{z}_i)\}$ for I_ε to be distinct, and to form a blow - up sequence when $\lambda_i \to 0$. Take $\mathbf{z}_i \;=\; V_{\lambda_i\,,\; \xi_i} \,\in\, \mathbf{Z}$.

§ 5 a. *Separation lemma.* Consider $V_{\lambda_o,\; \vec{0}}$, $V_{\lambda_1,\; \xi_1} \,\in\, \mathbf{Z}$, where $\lambda_1 \,<\, \lambda_o$. Via the parametrization (2.19), the pull-back metric (3.4) on $\mathbb{R}^+ \times \mathbb{R}^n$, and the identification with the (rescaled) hyperbolic metric in § 3 b, we find that

$$(5.1) \qquad \| V_{\lambda_o,\ \vec{0}} - V_{\lambda_1\,,\ \xi_1} \|_{\bigtriangledown} \ \geq \ \| V_{\lambda_o,\, 0} - V_{\lambda_1,\ \vec{0}} \|_{\bigtriangledown} = \ \int_{\lambda_1}^{\lambda_o} \frac{\bar{c}_o}{\lambda}\, d\lambda = \ \bar{c}_o \ln \left(\frac{\lambda_o}{\lambda_1} \right) .$$

Recall that vertical lines $(t\,,\ \bar{\xi})$ [with $\bar{\xi}$ fixed and $t \in \mathbb{R}^+$] are geodesics in the hyperbolic space. Thus the (curved) geodesic joining $(\lambda_o\,,\ \vec{0})$ and $(\lambda_1\,,\ \xi_1)$ in the hyperbolic space has longer length then that joining $(\lambda_o\,,\ \vec{0})$ and $(\lambda_1\,,\ \vec{0})$ whenever $\xi_1 \neq \vec{0}$.

Lemma 5.2. *Let*

$$(5.3) \qquad \mathbf{z}_o \ = \ V_{\lambda_o,\ \vec{0}} \quad \textit{and} \quad \mathbf{z}_1 \ = \ V_{\lambda_1\,,\ \xi_1}\,. \quad \textit{Assume that} \quad \frac{\lambda_o}{\lambda_1} \ \geq \ 2\,.$$

For $\bar{\varepsilon}_5$ small enough, and for all $|\,\varepsilon| \leq \bar{\varepsilon}_5$, we have

$$\mathbf{z}_o \,+\, w_\varepsilon(\mathbf{z}_o) \ \neq \ \mathbf{z}_1 \,+\, w_\varepsilon(\mathbf{z}_1)\,.$$

Here $w_\varepsilon(\mathbf{z}_o)$ and $w_\varepsilon(\mathbf{z}_1)$ are the solutions of the auxiliary equation (4.4) *as described in* Lemma 4.1.

Proof. Suppose the result is false. Then

$$\begin{aligned} & \mathbf{z}_o \,+\, w_\varepsilon(\mathbf{z}_o) \ = \ \mathbf{z}_1 \,+\, w_\varepsilon(\mathbf{z}_1) \\ \Longrightarrow \ & \bar{c}_o \ln 2 \ < \ \| V_{\lambda,\ \vec{0}} - V_{\lambda_1,\, \xi_1} \|_{\bigtriangledown} \ = \ \|w_\varepsilon(\mathbf{z}_1) - w_\varepsilon(\mathbf{z}_o)\|_{\bigtriangledown} \leq \| \, w_\varepsilon(\mathbf{z}_1)\|_{\bigtriangledown} + \| \, w_\varepsilon(\mathbf{z}_1)\|_{\bigtriangledown} \\ \Longrightarrow \ & \bar{c}_o \ln 2 \ \leq \ 2\,\bar{C}_4\,|\,\varepsilon|\,. \end{aligned}$$

Here we use (5.1) and the uniform bound in Lemma 4.16. Once we choose $\bar{\varepsilon}_5$ to be small enough, we have a contradiction. □

§ 5 b. *Blow - up lemma.*

Lemma 5.4. *Let $\{v_i\}_{i=1}^\infty$ be a sequence of positive $C^{2\,,\,\beta}$ solutions of equation* (2.5) *with $K \in C^{1,\ \tilde{\beta}}$. Suppose that*

$$(5.5) \qquad \| \, v_i - V_{\lambda_i,\ \xi_i} \|_{\bigtriangledown} \leq \, C\, \tilde{\varepsilon} \qquad \textit{for} \quad i \gg 1\,,$$

where $\xi_i \to \xi_o$ and $\lambda_i \to 0^+$. There is a positive number $\bar{\varepsilon}_6$ (depending on n only) such that for all δ small enough (depending on n and the sequence $\{v_i\}_{i=1}^\infty$), the sequence $\{v_i\}_{i=1}^\infty$ cannot stay bounded in $B_{\xi_o}(\delta)$, once (5.5) *is satisfied for $\tilde{\varepsilon} \ \leq \ \bar{\varepsilon}_6$.*

Proof. We compute

$$(5.6) \quad \int_{B_{\xi_o}(\delta)} |\bigtriangledown V_{\lambda_i\,,\ \xi_i}|^2 \ \geq \ \int_{B_{\xi_i}\left(\frac{\delta}{2}\right)} |\bigtriangledown V_{\lambda_i\,,\ \xi_i}|^2 \quad \text{for} \ \ i \gg 1 \ \text{so that} \ \ B_{\xi_i}(\delta/2) \subset B_{\xi_o}(\delta)$$

$$\geq \quad (n-2)^2 \int_{B_o\left(\frac{\delta}{2}\right)} \frac{\lambda_i^{n-2}\, r^2}{(\lambda_i^2 + r^2)^n}$$

[via a translation $(y - \xi_i) \to y$, find the gradient in polar coordinates]

$$= \quad \|S^{n-1}\| \cdot (n-2)^2 \int_0^{\frac{\delta}{2}} \frac{\lambda_i^{n-2}\, r^{n+1}}{(\lambda_i^2 + r^2)^n}\, dr$$

$$= \quad (n-2)^2 \int_0^{\arctan\left(\frac{\delta}{2\lambda_i}\right)} \frac{[\tan\theta]^{n+1}\ \sec^2\theta\, d\theta}{[\sec\theta]^{2n}} \qquad\qquad [r = \lambda_i \tan\theta]$$

$$= \quad (n-2)^2 \int_0^{\arctan\left(\frac{\delta}{2\lambda_i}\right)} [\sin\theta]^{n+1}\, [\cos\theta]^{n-3}\, d\theta \geq C_o\,, \quad \text{as long as} \quad \frac{\delta}{2\lambda_i} \geq 1\,.$$

Here C_o depends on n only. Suppose that $\{v_i\}_{i=1}^{\infty}$ is uniformly bounded in $B_{\xi_o}(\delta)$. Via equation (2.5) and elliptic regularity theory on gradient estimates [17], one infers that

$$(5.7) \qquad \|\nabla v_i\| \leq C_1 \quad \text{in} \quad B_{\xi_o}(\delta) \quad \text{for all} \quad i \gg 1\,.$$

Observe that the constant C_1 may depend on $\{v_i\}$. (5.6) leads to

$$
\begin{aligned}
(5.8) \qquad C_o \ &\leq \int_{B_{\xi_o}(\delta)} \|\nabla V_{\lambda_i,\xi_i}\|^2 = \int_{B_{\xi_o}(\delta)} \|\nabla v_i + (\nabla_o V_{\lambda_i,\xi_i} - \nabla v_i)\|^2 \\
&\leq 2 \int_{B_{\xi_o}(\delta)} \|\nabla v_i\|^2 + 2 \int_{B_{\xi_o}(\delta)} \|\nabla V_{\lambda_i,\xi_i} - \nabla v_i\|^2 \\
&\leq C_1^2 \cdot \frac{\|S^n\|}{n} \cdot \delta^n + 2\,\tilde{\varepsilon}^{\,2}\,.
\end{aligned}
$$

Combining (5.7) and (5.8), we obtain

$$C_o \leq C_1^2 \frac{\|S^n\|}{n} \cdot \delta^n + 2\,\tilde{\varepsilon}^{\,2}$$

Thus once we choose $\bar{\varepsilon}_5$ and δ so that

$$\frac{C_o}{2} > 2\,\bar{\varepsilon}_5 \quad \text{and} \quad \frac{C_o}{2} > C_1^2 \frac{\|S^n\|}{n} \cdot \delta^n\,,$$

we have a contradiction. Thus the conclusion of the lemma stands. □

§ 6. First order derivatives and the Kazdan - Warner condition.

From expressions (1.7) and (2.19), we obtain

$$
\begin{aligned}
(6.1) \qquad \frac{\partial G_{|\mathbf{z}}}{\partial \lambda}(\lambda,\, \xi) \ = \ \frac{n\,\bar{c}_{-1}}{\lambda} \Bigg\{ &\int_{\mathbb{R}^n} H(y) \left[\frac{\lambda^n}{(\lambda^2 + |\,y - \xi|^2)^n} \right] \\
&- 2 \int_{\mathbb{R}^n} H(y) \left[\frac{\lambda^{n+2}}{(\lambda^2 + |\,y - \xi|^2)^{n+1}} \right] \Bigg\}\,.
\end{aligned}
$$

This leads to

$$(6.2) \qquad \frac{\partial G_{|\mathbf{z}}}{\partial \lambda}(\lambda\,,\ \xi) \;=\; n\,\bar{c}_{-1}\,\lambda^{n-1}\int_{\mathbb{R}^n} H(y)\,\frac{1}{(\lambda^2+|\,y-\xi|^2)^n}\left[\frac{|\,y-\xi|^2-\lambda^2}{|\,y-\xi|^2+\lambda^2}\right] \;.$$

Likewise,

$$(6.3) \qquad \frac{\partial G_{|\mathbf{z}}}{\partial \xi_j}(\lambda\,,\ \xi) \;=\; -\,2n\,\bar{c}_{-1}\int_{\mathbb{R}^n} H(y)\,\frac{\lambda^n\,(\xi_j-y_j)}{(\lambda^2+|\,y-\xi|^2)^{n+1}}$$

for $j \;=\; 1,\ 2,\cdots,\ n$. Note that the integrals in (6.1) are independent on λ and ξ. This highlights the presence of λ^{-1} in (6.1). Cf. (4.17) and (4.27).

Lemma 6.4. *In* (6.3), *suppose* $\xi \;=\; 0$, *and* $H\,(y) \;=\; H\,(|\,y|)$ *is radially symmetric and bounded, then*

$$\frac{\partial G_{|\mathbf{z}}}{\partial \xi_j}(\lambda\,,0) \;=\; 0 \qquad \textit{for} \quad \lambda > 0 \quad \textit{and} \quad 1 \le j \le n\,.$$

Proof. This is because the integrals

$$\int_{\mathbb{R}^n} H\,(|\,y|)\cdot\frac{y_j}{(\lambda^2+|\,y|^2)^{n+1}} \qquad (j \;=\; 1,\ 2,\cdots,\ n)$$

vanish, due to antisymmetry. □

§ 6 a. *Geometric interpretation.* Via the change of variables $\lambda\,\bar{y} \;=\; y-\xi$, we have

$$(6.5) \qquad \frac{\partial G_{|\mathbf{z}}}{\partial \lambda}(\lambda\,,\xi) \;=\; \frac{1}{2^n}\cdot\frac{n\,\bar{c}_{-1}}{\lambda}\left[\int_{\mathbb{R}^n} H(\lambda\,\bar{y}\,+\,\xi)\left(1-\frac{2}{1+|\,\bar{y}|^2}\right)\frac{2^n}{(1+|\,\bar{y}|^2)^n}\,d\bar{y}\right]$$

$$\Longrightarrow \quad \frac{\partial G_{|\mathbf{z}}}{\partial \lambda}(\lambda\,,\xi) \;=\; \frac{1}{\lambda}\cdot\frac{n\,\bar{c}_{-1}}{2^n}\int_{S^n} H(\lambda\,\mathcal{P}(x)\,+\,\xi)\,x_{n+1}\,dS_x \qquad [\,\text{via}\ \ (2.2)\ \&\ \ (2.3)\,]$$

$$= \;\frac{1}{\lambda}\cdot\frac{n\,\bar{c}_{-1}}{2^n}\cdot\left[\int_{S^n_+} H(\lambda\,\mathcal{P}(x)\,+\,\xi)\,x_{n+1}\,dS_x \;+\; \int_{S^n_-} H(\lambda\,\mathcal{P}(x)\,+\,\xi)\,x_{n+1}\,dS_x\right]$$

Intuitively, one can say that a critical point is a balance between the upper (S^n_+) and lower (S^n_-) hemisphere. Likewise,

$$(6.6) \qquad \frac{\partial G_{|\mathbf{z}}}{\partial \xi_j}(\lambda\,,\xi) \;=\; 2n\,\bar{c}_{-1}\int_{\mathbb{R}^n} H(y)\,\frac{\lambda^n\,(y_j-\xi_j)}{(\lambda^2+|\,y-\xi|^2)^{n+1}}$$

$$= \;\frac{n\,\bar{c}_{-1}}{\lambda}\int_{\mathbb{R}^n} H(\lambda\,\bar{y}+\xi)\left(\frac{2\,\bar{y}_j}{1+|\,\bar{y}|^2}\right)\frac{1}{(1+|\,\bar{y}|^2)^n}$$

$$= \;\frac{n\,\bar{c}_{-1}}{\lambda}\int_{\mathbb{R}^n} H(\lambda\,\bar{y}+\xi)\left(\frac{2\,\bar{y}_j}{1+|\,\bar{y}|^2}\right)\frac{1}{(1+|\,\bar{y}|^2)^n}$$

$$\Longrightarrow \quad \frac{\partial G_{|\mathbf{z}}}{\partial \xi_j}(\lambda\,,\xi) \;=\; \frac{1}{\lambda}\cdot\frac{n\,\bar{c}_{-1}}{2^n}\int_{S^n} H(\lambda\,\mathcal{P}(x)+\xi)\,x_j\,dS_x \qquad [\,\text{via}\ \ (2.2)\ \&\ \ (2.3)\,]$$

for $j = 1, 2, \cdots, n$. It follows that if $\mathcal{H}$ is antipodal symmetric, i.e., $\mathcal{H}(-x) = \mathcal{H}(x)$, then $(1, \vec{0})$ is a critical point of $G_{|\mathbf{Z}}$. [Recall that $\mathcal{H}$ and H are related via (2.6).]

Next, we observe that $x_1, \cdots, x_{n+1}$ are first eigenfunctions of the standard Laplacian on S^n. Using (6.5) and (6.5), we have the following.

Proposition 6.7. *Let $\mathcal{H}$ and H be related by (2.6). Suppose that $\mathcal{H}(x)$ is orthogonal (with respect to the L^2-inner product) to the first eigenspace of Δ_1. Then $(1, \vec{0})$ is a critical point of $G_{|\mathbf{Z}}$.*

§ 6 b. *Homogeneous harmonic polynomials.* In order to apply Proposition 6.7 we consider in particular order k homogeneous harmonic polynomials P^k_Δ on $\mathbb{R}^{n+1}$, where $k > 1$ is an integer. It is well known that

$$\Delta_1 P^k_\Delta + k(k+n-1)P^k_\Delta = 0 \quad \text{in } S^n .$$

For $k > 1$, each $\mathcal{H} = P^k_\Delta$ is orthogonal to the first eigenspace, i.e., the condition in Proposition 6.7 is fulfilled.

§ 6 c. *Kazdan - Warner condition.* A deep relation is revealed when we differentiate $\mathcal{K}$ with respect to a *conformal Killing vector field* $\mathbf{X}$ (generating a family of conformal transformations). In this way we obtain the renowned Kazdan - Warner formula [7]

$$\int_{S^n} \mathbf{X}(\mathcal{K})\, u^{\frac{2n}{n-2}}\, dS = 0 \qquad [\text{refer to equation (1.1), here } \mathbf{X}(\mathcal{K}) = \langle \mathbf{X} \;\; \triangledown \mathcal{K}\rangle_{g_1}] .$$

This is a necessary (but not sufficient) condition on $\mathcal{K}$ for equation (1.1) has a positive solution (see [18]).

Definition 6.8. *A function $\mathcal{K} \in C^1(S^n)$ is said to satisfy the K-W condition if there exists a (single) positive function $\psi \in C^o(S^n)$ such that*

$$\text{(6.9)} \qquad \int_{S^n} \mathbf{X}(\mathcal{K})\, \psi^{\frac{2n}{n-2}}\, dS = 0 \qquad \textit{for all conformal Killing vector fields } \mathbf{X} .$$

Theorem 6.10. *For a given $\mathcal{H} \in C^1(S^n)$, suppose $G_{|\mathbf{Z}}$ has a critical point at (λ_c, ξ_c), then*

$$\mathcal{K} = 4n(n-1) + \varepsilon\, \mathcal{H}$$

fulfills the $K-W$ conditions (6.9) for any $\varepsilon \in \mathbb{R}$, with

$$\psi(x) = \left(\frac{\lambda_c}{\lambda_c^2 + |\, y - \xi_c|^2} \right)^{\frac{n-2}{2}} \cdot \left(1 + |\, y|^2\right)^{\frac{n-2}{2}} \qquad \textit{for} \quad y = \dot{\mathcal{P}}(x) .$$

As the proof of Theorem 6.10 is well-known, we direct the interested readers to § A.11 in the e-Appendix for reference. Nevertheless, Theorem 6.10 highlights the balance achieved by a critical point of $G_{|\mathbf{z}}$.

§ 7. Second derivatives, symmetry and stability conditions.

We continue from (6.2) – (6.3),

$$\frac{\partial^2 G_{|\mathbf{z}}}{\partial \xi_j \, \partial \xi_\ell}(\lambda, \xi) = 2^2 \, n \, (n+1) \, \bar{c}_{-1} \int_{\mathbb{R}^n} H(y) \cdot \frac{\lambda^n \, (\xi_j - y_j)(\xi_\ell - y_\ell)}{(\lambda^2 + |y - \xi|^2)^{n+2}} \, , \tag{7.1}$$

$$\frac{\partial^2 G_{|\mathbf{z}}}{\partial \lambda \, \partial \xi_\ell}(\lambda, \xi) = -\, 2n^2 \, \bar{c}_{-1} \int_{\mathbb{R}^n} H(y) \cdot \frac{\lambda^{n-1} \, (\xi_\ell - y_\ell)}{(\lambda^2 + |y - \xi|^2)^{n+1}} \; + \tag{7.2}$$

$$+ \, 2n \, (n+1) \, \bar{c}_{-1} \int_{\mathbb{R}^n} H(y) \cdot \frac{2 \, \lambda^{n+1} \, (\xi_\ell - y_\ell)}{(\lambda^2 + |y - \xi|^2)^{n+2}} \quad \text{for } \; 1 \le j \ne \ell \le n \, .$$

Moreover,

$$\frac{\partial^2 G_{|\mathbf{z}}}{\partial \xi_\ell^2}(\lambda, \xi) = -\, 2n \, \bar{c}_{-1} \int_{\mathbb{R}^n} H(y) \cdot \frac{\lambda^n}{(\lambda^2 + |y - \xi|^2)^{n+1}} + 4n \, (n+1) \, \bar{c}_{-1} \int_{\mathbb{R}^n} H(y) \cdot \frac{\lambda^n \, (\xi_\ell - y_\ell)^2}{(\lambda^2 + |y - \xi|^2)^{n+2}} \tag{7.3}$$

$$\Longrightarrow \quad \frac{\partial^2 G_{|\mathbf{z}}}{\partial \xi_\ell^2}(\lambda, 0) = -\, 2 \, n \, \bar{c}_{-1} \int_{\mathbb{R}^n} H(y) \cdot \frac{\lambda^n}{(\lambda^2 + |y|^2)^{n+1}} + 4 \, n \, (n+1) \, \bar{c}_{-1} \int_{\mathbb{R}^n} H(y) \cdot \frac{\lambda^n \, (y_\ell)^2}{(\lambda^2 + |y|^2)^{n+2}} \, .$$

It follows from (6.1) that

$$\begin{aligned} \frac{\partial^2 G_{|\mathbf{z}}}{\partial \lambda^2}(\lambda, \xi) = \; & n \, (n-1) \, \bar{c}_{-1} \int_{\mathbb{R}^n} H(y) \cdot \frac{\lambda^{n-2}}{(\lambda^2 + |y - \xi|^2)^n} \\ & -\, 2n^2 \, \bar{c}_{-1} \int_{\mathbb{R}^n} H(y) \cdot \frac{\lambda^n}{(\lambda^2 + |y - \xi|^2)^{n+1}} \\ & -\, 2n \, (n+1) \, \bar{c}_{-1} \int_{\mathbb{R}^n} H(y) \cdot \frac{\lambda^n}{(\lambda^2 + |y - \xi|^2)^{n+1}} \\ & +\, 4 \, n \, (n+1) \, \bar{c}_{-1} \int_{\mathbb{R}^n} H(y) \cdot \frac{\lambda^{n+2}}{(\lambda^2 + |y - \xi|^2)^{n+2}} \, . \end{aligned} \tag{7.4}$$

§ 7 a. *Second derivative at a critical point.* Let (λ_c, ξ_c) be a critical point for $G_{|\mathbf{z}}$. From (7.1), (7.2) and (6.3) we obtain

$$\frac{\partial^2 G_{|\mathbf{z}}}{\partial \xi_j \, \partial \xi_\ell}(\lambda_c, \xi_c) = 4 \, n \, (n+1) \, \bar{c}_{-1} \int_{\mathbb{R}^n} H(y) \cdot \frac{\lambda_c^n \, (\xi_{cj} - y_j)(\xi_{c\ell} - y_\ell)}{(\lambda^2 + |y - \xi_c|^2)^{n+2}} \, , \tag{7.5}$$

$$(7.6) \qquad \frac{\partial^2 G_{|\mathbf{z}}}{\partial\lambda\,\partial\,\xi_\ell}\,(\,\lambda_c\,,\,\xi_c) \;=\; 2\,n\,(n+1)\,\bar{c}_{-1}\int_{\mathbb{R}^n} H(y)\cdot\frac{\lambda_c^{n+1}\,(\xi_{c\ell}-y_\ell)}{(\,\lambda_c^2+|\,y-\xi_c|^{\,2})^{n+2}}\;.$$

Here $1\,\le\,j\,\ne\,\ell\,\le\,n\,.$ Before we continue, we register the following observation.

Lemma 7.7. *In (7.5) and (7.6), suppose $\xi_c = 0$ and $H(y) = H(|\,y|)$ is radially symmetric, then*

$$\frac{\partial^2 G_{|\mathbf{z}}}{\partial\,\xi_j\,\partial\,\xi_\ell}\,(\,\lambda_c\,,\,0) \;=\; 0 \;=\; \frac{\partial^2 G_{|\mathbf{z}}}{\partial\lambda\,\partial\,\xi_\ell}\,(\,\lambda_c\,,\,0) \qquad \text{for} \quad 1\,\le\,j\,\ne\,\ell\,\le\,n\,.$$

That is, the Hessian matrix is diagonal.

Proof. This is because the integrals

$$\int_{\mathbb{R}^n} H\,(|\,y|)\cdot\frac{y_j\,y_\ell}{(\,\lambda^2+|\,y|^{\,2})^{n+2}} \qquad (j\,\ne\,\ell) \quad \text{and} \quad \int_{\mathbb{R}^n} H\,(|\,y|)\cdot\frac{y_\ell}{(\,\lambda^2+|\,y|^{\,2})^{n+2}}$$

vanish by antisymmetry argument. □

From (7.4), together with (6.1), we obtain

$$\begin{aligned}
(7.8) \quad \frac{\partial^2 G_{|\mathbf{z}}}{\partial\lambda^2}(\,\lambda_c\,,\,\xi_c) \;&=\; 2\,n\,(n-1)\,\bar{c}_{-1}\int_{\mathbb{R}^n} H(\,y)\cdot\frac{\lambda_c^n}{(\,\lambda_c^2+|\,y-\xi_c|^{\,2})^{n+1}} \\
&\quad -\,2\,n^2\;\bar{c}_{-1}\int_{\mathbb{R}^n} H(\,y)\cdot\frac{\lambda_c^n}{(\,\lambda_c^2+|\,y-\xi_c|^{\,2})^{n+1}} \\
&\quad -\,2\,n\,(n+1)\;\bar{c}_{-1}\int_{\mathbb{R}^n} H(\,y)\cdot\frac{\lambda_c^n}{(\,\lambda_c^2+|\,y-\xi_c|^{\,2})^{n+1}} \\
&\quad +\,4\,n\,(n+1)\;\bar{c}_{-1}\int_{\mathbb{R}^n} H(\,y)\cdot\frac{\lambda_c^{n+2}}{(\,\lambda_c^2+|\,y-\xi_c|^{\,2})^{n+2}} \\
&=\; -2\,n\,(n+2)\,\bar{c}_{-1}\int_{\mathbb{R}^n} H(\,y)\cdot\frac{\lambda_c^n}{(\,\lambda_c^2+|\,y-\xi_c|^{\,2})^{n+1}} \\
&\quad +\,4\,n\,(n+1)\,\bar{c}_{-1}\int_{\mathbb{R}^n} H(\,y)\cdot\frac{\lambda_c^{n+2}}{(\,\lambda_c^2+|\,y-\xi_c|^{\,2})^{n+2}}\;.
\end{aligned}$$

[Check: $2n\,(n-1)-2n^2-2n\,(n+1) \;=\; -2n\,(n+2)\,.$] As for the second derivatives in $\xi_\ell\,,$ from (7.3), we have

$$\begin{aligned}
(7.9) \qquad \sum_{\ell=1}^{n}\frac{\partial^2 G_{|\mathbf{z}}}{\partial\,\xi_\ell^2}\,(\,\lambda_c\,,\;\xi_c) \;&=\; -\,2n^2\,\bar{c}_{-1}\int_{\mathbb{R}^n} H(\,y)\cdot\frac{\lambda_c^n}{(\,\lambda_c^2+|\,y-\xi_c|^{\,2})^{n+1}} \\
&\quad +\,4\,n\,(n+1)\,\bar{c}_{-1}\int_{\mathbb{R}^n} H(\,y)\cdot\frac{\lambda_c^n\,\sum_{\ell=1}^n(\xi_{c\ell}-y_\ell)^2}{(\,\lambda_c^2+|\,y-\xi_c|^{\,2})^{n+2}} \\
&=\; -\,2\,n^2\,\bar{c}_{-1}\int_{\mathbb{R}^n} H(y)\cdot\frac{\lambda_c^n}{(\,\lambda_c^2+|\,y-\xi_c|^{\,2})^{n+1}}
\end{aligned}$$

$$
\begin{aligned}
& \quad + 4\, n(n+1)\, \bar{c}_{-1} \int_{\mathbb{R}^n} H(y) \cdot \frac{\lambda_c^n \,|\, y - \xi_c|^{\,2}}{(\,\lambda_c^2 + |\, y - \xi_c|^{\,2})^{n+2}} \\
= & \quad - 2\, n^2\, \bar{c}_{-1} \int_{\mathbb{R}^n} H(y) \cdot \frac{\lambda_c^n}{(\,\lambda_c^2 + |\, y - \xi_c|^{\,2})^{n+1}} \\
& \quad + 4\, n(n+1)\, \bar{c}_{-1} \int_{\mathbb{R}^n} H(y) \cdot \frac{\lambda_c^n\, (\,\lambda_c^2 + |\, y - \xi_c|^{\,2})}{(\,\lambda_c^2 + |\, y - \xi_c|^{\,2})^{n+2}} \\
& \quad - 4\, n(n+1)\, \bar{c}_{-1} \int_{\mathbb{R}^n} H(y) \cdot \frac{\lambda_c^n \times \lambda_c^2}{(\,\lambda_c^2 + |\, y - \xi_c|^{\,2})^{n+2}} \\
= & \quad - 2\, n^2\, \bar{c}_{-1} \int_{\mathbb{R}^n} H(y) \cdot \frac{\lambda_c^n}{(\,\lambda_c^2 + |\, y - \xi_c|^{\,2})^{n+1}} \\
& \quad + 4\; n(n+1)\, \bar{c}_{-1} \int_{\mathbb{R}^n} H(y) \cdot \frac{\lambda_c^n}{(\,\lambda_c^2 + |\, y - \xi_c|^{\,2})^{n+1}} \\
& \quad - 4\, n\,(n+1)\, \bar{c}_{-1} \int_{\mathbb{R}^n} H(y) \cdot \frac{\lambda_c^n \times \lambda_c^2}{(\,\lambda_c^2 + |\, y - \xi_c|^{\,2})^{n+2}} \\
= & \quad 2\, n\,(n+2)\, \bar{c}_{-1} \int_{\mathbb{R}^n} H(y) \cdot \frac{\lambda_c^n}{(\,\lambda_c^2 + |\, y - \xi_c|^{\,2})^{n+1}} \\
& \quad - 4\, n\,(n+1)\, \bar{c}_{-1} \int_{\mathbb{R}^n} H(y) \cdot \frac{\lambda_c^{n+2}}{(\,\lambda_c^2 + |\, y - \xi_c|^{\,2})^{n+2}} \quad .
\end{aligned}
$$

[Check: $4n\,(n+1) - 2n^2 \;=\; 2n^2 + 4n\,.$] Combining (7.8) and (7.9), we observe the following result.

Proposition 7.10. *At a critical point* $(\,\lambda_c\,,\ \xi_c)$ *for* $G_{|\mathbf{z}}\,,$ *the Hessian matrix is trace-free. That is,*

$$
\frac{\partial^2 G_{|\mathbf{z}}}{\partial \lambda^2}\,(\,\lambda_c\,,\ \xi_c) \;=\; -\left[\,\sum_{\ell\,=\,1}^{n} \frac{\partial^2 G_{|\mathbf{z}}}{\partial\, \xi_\ell^2}\,(\,\lambda_c\,,\ \xi_c)\right]. \tag{7.11}
$$

It is well-known that at a critical point, the Hessian matrix being positive definite (resp. negative definite) is a sufficient condition for local minimum (resp. local maximum). In some sense, Proposition 7.10 guides us to look for saddle points.

§ 7 b. *Geometric interpretation of the second derivatives.* If $(1,\ \vec{0}\,)$ is a critical point, then

$$
\begin{aligned}
\frac{\partial^2 G_{|\mathbf{z}}}{\partial\, \xi_j\, \partial\, \xi_\ell}\,(1,\ \vec{0}\,) \;&=\; 4\, n\,(n+1)\, \bar{c}_{-1} \int_{\mathbb{R}^n} H(y) \cdot \frac{y_\ell\, y_j}{(1 + |\, y|^{\,2})^{n+2}} \\
&=\; \frac{n\,(n+1)}{2^n}\, \bar{c}_{-1} \int_{S^n} x_\ell\, x_j\, H(x)\, dS_x \qquad (1\,\le\, i \,\ne\, j\,\le\, n)\,,
\end{aligned} \tag{7.12}
$$

$$(7.13) \qquad \frac{\partial^2 G_{|\mathbf{z}}}{\partial\lambda\,\partial\,\xi_\ell}(1,\ \vec{0}\,) \;=\; 2\,n\,(n+1)\,\bar{c}_{-1}\int_{\mathbb{R}^n} H(y)\cdot\frac{(-y_\ell)}{(1+|\,y|^{\,2})^{n+2}}$$

$$= \; -\,\frac{n\,(n+1)}{2^{n+1}}\,\bar{c}_{-1}\int_{S^n} x_\ell\,(1-x_{n+1})\,H(x)\,dS_x\,,$$

$$(7.14) \quad \frac{\partial^2 G_{|\mathbf{z}}}{\partial\lambda^2}(1,\ \vec{0}\,) \;=\; \frac{n\,(n+1)}{2^n}\cdot\bar{c}_{-1}\left[\int_{S^n} x_{n+1}^2\,H(x)\,dS_x \;-\; \frac{1}{n+1}\int_{S^n} H(x)\,dS_x\right],$$

$$(7.15) \quad \frac{\partial^2 G_{|\mathbf{z}}}{\partial\,\xi_\ell^2}(1,\ \vec{0}\,) \;=\; -\,\frac{n}{2^n}\cdot\bar{c}_{-1}\int_{S^n} H(x)\,dS_x \;+\; \frac{n\,(n+1)}{2^n}\cdot\bar{c}_{-1}\int_{S^n}(x_\ell)^2\,H(x)\,dS_x\,.$$

We show in § A.11 the calculations in obtaining (7.14) and (7.15). Observe that (7.11) is fulfilled once we combine (7.14) and (7.15) with

$$x_1^2+\cdots+x_n^2 \;=\; 1-x_{n+1}^2 \quad \text{on } \ S^n.$$

§ 7 c. *Symmetry.* Consider the symmetry described by

$$(7.16) \quad H\,(y_1,\cdots,\ -y_\ell\,,\ \cdots,\ y_n) \;=\; H\,(y_1,\cdots,\ y_\ell\,,\ \cdots,\ y_n) \quad \text{for } \ \ell \;=\; 1,\,2,\cdots,\ n\,,$$

$$(7.17) \qquad H\,(r,\ \vartheta) \;=\; H\left(\frac{1}{r}\,,\ \vartheta\right) \qquad \text{for } \ r\in(0,\ 1) \quad \text{and } \ \vartheta \;:=\; \frac{y}{|\,y|}\in S^{n-1}.$$

Here $(r\,,\ \vartheta)$ is the polar coordinates of $\mathbb{R}^n$. In this case H is determined by its values in the 'truncated quarter'

$$\{\,y \;=\; (y_1,\ \cdots,\ y_n)\in\mathbb{R}^n \;\mid\; y_1\ge 0\,,\cdots,\ y_n\ge 0 \;\;\&\;\; |y| \;\le 1\,\}\,.$$

Theorem 7.18. *Assume that* $H\in C^1(\mathbb{R}^n)$ *satisfies the symmetric conditions* (7.16), (7.17), *and the stability conditions*

$$(7.19) \quad \int_{\mathbb{R}^n} H(y)\cdot\left(\frac{1}{1+|\,y|^{\,2}}\right)^{n+1} \;\ne\; 2\,(n+1)\int_{\mathbb{R}^n} H(y)\cdot y_\ell^2\left(\frac{1}{1+|\,y|^{\,2}}\right)^{n+2}$$

for $\ell \;=\; 1,\,2,\cdots,\ n\,,$ *and*

$$(7.20) \quad \int_{\mathbb{R}^n} H(y)\cdot\left(\frac{1}{1+|\,y|^{\,2}}\right)^{n+1} \;\ne\; \frac{2\,(n+1)}{n+2}\int_{\mathbb{R}^n} H(y)\cdot\left(\frac{1}{1+|\,y|^{\,2}}\right)^{n+2}\,.$$

Then equation (1.5) *with relation* (1.6) *has a positive* C^2 *solution when* $|\,\varepsilon|$ *is small.*

Proof. Refer to equation (6.2) with $\lambda = 1$ and $\xi = 0$. Using a change of variables, we obtain

$$\begin{aligned}
& n\,\bar c_{-1} \int_{{\rm I\!R}^n \setminus B_o(1)} H(y)\cdot \frac{1}{(1+r^2)^n}\left[\frac{r^2-1}{r^2+1}\right] \qquad\qquad (r = |\,y|\,) \\
= \quad & n\,\bar c_{-1}\,\|\,S^{n-1}\| \int_{S^{n-1}}\int_1^\infty H(r,\ \theta)\cdot \frac{1}{(1+r^2)^n}\left[\frac{r^2-1}{r^2+1}\right] r^{n-1}\,dr\,dS_\theta \\
= \quad & -\,n\,\bar c_{-1}\,\|\,S^{n-1}\| \int_{S^{n-1}}\int_0^1 H\left(\frac{1}{\bar r}\ ,\ \theta\right)\cdot \left[\frac{|\,\bar r|^2-1}{|\,\bar r|^2+1}\right]\left(\frac{1}{1+|\,\bar r|^2}\right)^n \bar r^{n-1}\ d\bar r\,dS_\theta \\
\Longrightarrow \quad & \frac{\partial G_{|\mathbf z}}{\partial \lambda}(1,\ \vec 0\,) \qquad\qquad (\,\text{here } \bar r = r^{-1}\,)
\end{aligned}$$

$$= n\,\bar c_{-1}\,\|\,S^{n-1}\| \int_{S^{n-1}}\int_0^1 \left[H(\,s\,,\ \theta) - H\left(\frac{1}{s}\ ,\ \theta\right)\right]\left[\frac{1-s^2}{1+s^2}\right]\left(\frac{1}{1+\,s^2}\right)^n s^{n-1} ds\,dS_\theta = 0$$

(via condition (7.17)).

In view of (7.16), we also have

$$\frac{\partial G_{|\mathbf z}}{\partial\,\xi_i}(1,\ \vec 0\,) = 2\,n\,\bar c_{-1}\int_{{\rm I\!R}^n}\left[\frac{1}{(1+|\,y|^2)^{n+1}}\right][H(y)\cdot y_i] = 0\,.$$

That is, $(1,\ \vec 0\,)$ is a critical point of $G_{|\mathbf z}$. Consider the Hessian matrix at $(1,\ \vec 0\,)$. From (7.12), (7.13) and (7.16) we see that

$$\begin{aligned}
\frac{\partial^2 G_{|\mathbf z}}{\partial\,\xi_j\,\partial\,\xi_\ell}(1,\ \vec 0\,) &= 2^2\,n\,(n+1)\,\bar c_{-1}\int_{{\rm I\!R}^n}\frac{1}{(1+|\,y|^2)^{n+2}}[H(y)\cdot y_\ell\, y_j] = 0 \\
\frac{\partial^2 G_{|\mathbf z}}{\partial\lambda\,\partial\,\xi_\ell}(1,\ \vec 0\,) &= 2n^2\,\bar c_{-1}\int_{{\rm I\!R}^n}\frac{1}{(1+|\,y|^2)^{n+1}}\,[\,H(y)\cdot y_\ell\,] \\
&\quad -\,4\,n\,(n+1)\,\bar c_{-1}\int_{{\rm I\!R}^n}\frac{1}{(1+|\,y|^2)^{n+2}}\,[\,H(y)\cdot y_\ell\,] = 0\,.
\end{aligned}$$

Here we apply symmetry argument for $1 \le j \ne \ell \le n$. From (7.19) and (7.20), together with the first equalities in (7.14) and (7.15), we obtain

$$\frac{\partial^2 G_{|\mathbf z}}{\partial\lambda^2}(1,\ \vec 0\,) \ne 0 \quad \text{and} \quad \frac{\partial^2 G_{|\mathbf z}}{\partial\,\xi_\ell^2}(1,\ \vec 0\,) \ne 0 \quad \text{for } \ell = 1,\ 2,\cdots,\ n\,.$$

That is, $(1,\ \vec 0\,)$ is a stable critical point of $G_{|\mathbf z}$. Using Theorem 2.17 in [4], pp. 25, together with elliptic regularity theory, we obtain a positive C^2 solution of equation (1.5) with relation (1.6) for $|\,\varepsilon|$ small enough. □

e-Appendix can be found in www.math.nus.edu.sg/˜matlmc/e-Appendix.pdf

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Construction of Blow-up Sequences for the Prescribed Scalar Curvature Equation on S^n. II. Annular Domains

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Abstract

Using the Lyapunov - Schmidt reduction method, we describe how to use annular domains to construct (scalar curvature) functions on S^n $(n \ge 6)$, so that each one of them enables the conformal scalar curvature equation to have a blowing - up sequence of positive solutions. The prescribed scalar curvature function is shown to have $C^{n-1,\,\beta}$ smoothness.

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1. Introduction.

In this article we apply the Lyapunov - Schmidt reduction method to find non - constant functions $\mathcal{K}$ so that the equation

$$(1.1) \qquad \Delta_1 u - \tilde c_n\, n\,(n-1)\, u \; + \; (\tilde c_n\, \mathcal{K}) u^{\frac{n+2}{n-2}} \;=\; 0 \quad \text{in} \;\; S^n$$

has infinite number of positive solutions, which compose a blow - up sequence of solutions. We refer to § 1 c for the (rather standard) notations we use.

As often with good insight, the Lyapunov - Schmidt reduction method works with miraculous simplicity. It is used successfully by Ambrosetti, Berti and Malchiodi to construct many C^k - metrics g on S^n such that the Yamabe equation

$$(1.2) \qquad \Delta_g\, u \; - \; \tilde c_n R_g\, u \; + \; \tilde c_n\, n\,(n-1)\, u^{\frac{n+2}{n-2}} \;=\; 0 \qquad \text{in} \quad S^n$$

has infinite number of positive solutions [here g is non - conformal to g_1 – the standard metric on S^n ; R_g is the scalar curvature of $(S^n,\, g)$] . See the renowned monograph [3] , in which we can find their work on equation (1.1) as well. Initially, the degree of smoothness on g is restricted to $2 \le k \le (n-3)/4$, but is improved to C^∞ by Brendle in [4] for $n \ge 52$, and together with Marques [5] for the remaining cases $25 \le n \le 51$. The result should be read with [10] , in which compactness theorem for the Yamabe equation is obtained for $n \le 24$ (see also [18], [19] and [20] for earlier results). The unexpected critical dimension $n = 24$ is both fascinating and intriguing.

It is rewarding to view the prescribed scalar curvature equation (1.1) as a kind of "dual" to the Yamabe equation (1.2). In contrast, blow - up sequences of positive solutions for (1.1) are widely studied as a mean to find solutions. However, to the knowledge of the author, no general method is known to construct blow - up sequences of positive solutions for equation (1.1) [that is, in S^n , with fixed critical index $(n+2)/(n-2)$ and fixed $\mathcal{K}$] . Apparently the only previously known case is when $\mathcal{K}$ is equal to a positive constant. With the method presented in this article, we are able to generate simple blow - ups (see [10] and [17]). The solutions are perturbations of the standard ones [cf. (1.4) below], and are adjusted to maintain the same scalar curvature function $\mathcal{K}$. The degree of flatness of $\mathcal{K}$ at the blow - up point (the south pole) is up to $(n-1)$ [refer to § 3 g and (3.30)] .

Recalling the notations used in Part I [16] , the reduced functional for equation (1.1) is defined by

$$(1.3) \quad G_{|\mathbf{z}}(\mathbf{z}) \;=\; \bar c_{-1} \int_{\mathbb{R}^n} H(y) \left[\frac{\lambda}{\lambda^2 + |\, y - \xi|^2} \right]^n dy \qquad \text{for} \quad \mathbf{z} = V_{\lambda,\,\xi} \in \mathbf{Z}\,, \qquad \text{where}$$

$$(1.4) \quad \mathbf{Z} \;:=\; \left\{ \, V_{\lambda,\,\xi}\,(y) \;:=\; \left(\frac{\lambda}{\lambda^2 + |\, y - \xi|^2} \right)^{\frac{n-2}{2}} \quad \text{with} \quad (\lambda,\; \xi) \in \mathbb{R}^+ \times \mathbb{R}^n \right\},$$

$$(1.5) \qquad K(y) \;:=\; \mathcal{K}\,(\dot{\mathcal{P}}^{-1}(y)) \;=\; 4\,n\,(n-1) \;+\; \varepsilon\,H(y) \qquad \text{for} \quad y \in \mathbb{R}^n\,.$$

In (1.3), $\bar c_{-1} \;=\; -\,[\,\tilde c_n \cdot (n-2)]\,/(2n)$. For $\varepsilon \in \mathbb{R}$ small, the question on finding a positive solution of equation (1.1) is reduced to finding a *stable* critical point of $G_{|\mathbf{z}}$. See [3] or [16]. It is shown in Part I [16] (Proposition 7.10), that the Hessian matrix of $G_{|\mathbf{z}}$ at a critical point is always trace-free, thus can never be positive or negative definite. Instead of looking for strict local maximum or minimum (as in the Yamabe equation), we seek *saddle points*. Although it is a finite dimensional problem, practical examples show that it can be illusive to locate (local) critical points. Indeed, the existence of a critical point implies the fulfillment of the Kazdan-Warner condition for $\mathcal{K}$ (see Theorem 6.10 in [16] for full details).

In [1], using the Lyapunov-Schimdt method and a degree counting method, (finite number of) stable critical points of the $G_{|\mathbf{z}}$ are found. The research is carried on in [2], where the authors explore symmetries. Each of these methods does not readily juxtapose to produce a blow-up sequence of solutions for the scalar curvature equation (1.1).

In this article, fixing an annular domain (see § 1 a below), we determine precisely the critical point of $G_{|\mathbf{z}}$, and show that the Hessian matrix at the critical point is non-degenerate (hence a stable critical point). By superimposing concentric annular domains, and carefully estimating the gradient interference, we are able to find infinite number of stable critical points via degree theory for maps. With the uniform cancelation property in the Lyapunov-Schmidt method (described in Part I [16]), a blow-up sequence can be found. We obtain the claimed $C^{\,n-1,\,\beta}$ regularity in § 3 g and § 3 h by choosing the "strength" [see (1.6)] and the radii of the annular domains suitable. Regarding the degree of smoothness, cf. also Remark 7.11 in [14]. We observe that similar argument works for slightly offset, non-concentric annular domains.

The search for a construction on higher energy blow-ups (e.g. towering blow-ups, as well as the more complicated aggregated and clustered blow-ups) is a challenging project for research. See [15] concerning a classification of blow-ups for (1.1) (cf. also [17] and [21]). Refer to [12] [13] for non-compact spaces.

§ 1 a. *Main Result.* Consider the annular domains

$$B_o\left(\frac{1+\eta}{\mathbf{a}}\right) \Big\backslash\, \overline{B_o\left(\frac{1-\eta}{\mathbf{a}}\right)}\,, \quad \cdots\,, \quad B_o\left(\frac{1+\eta}{\mathbf{a}^k}\right) \Big\backslash\, \overline{B_o\left(\frac{1-\eta}{\mathbf{a}^k}\right)}\,, \quad \cdots\,.$$

We give explicit conditions on the numbers $\mathbf{a} > 1$ and $\eta \in (0,\,1)$ in (1.9). Choose a small positive number σ and fix a number $\tau \in (n-1\,,\ n)$. Let H^S be given by

$$(1.6) \qquad H^S\,(y) \;=\; \begin{cases} \dfrac{1}{\mathbf{a}^{\tau k}} & \text{if} \quad y \in B_o\left(\dfrac{1\,+\,\eta}{\mathbf{a}^k}\right) \Big\backslash\, B_o\left(\dfrac{1\,-\,\eta}{\mathbf{a}^k}\right), \qquad k = 1,\ 2, \cdots, \\[2ex] 0 & \text{if} \quad y \not\in \bigcup_{k=1}^{\infty} \left\{ B_o\left(\dfrac{1\,+\,(\eta\,+\,\sigma)}{\mathbf{a}^k}\right) \Big\backslash\, B_o\left(\dfrac{1\,-\,(\eta\,+\,\sigma)}{\mathbf{a}^k}\right) \right\}. \end{cases}$$

In between $B_o\left(\frac{1+(\eta+\sigma)}{\mathbf{a}^k}\right)\Big\backslash B_o\left(\frac{1+\eta}{\mathbf{a}^k}\right)$ and $B_o\left(\frac{1-\eta}{\mathbf{a}^k}\right)\Big\backslash B_o\left(\frac{1-(\eta+\sigma)}{\mathbf{a}^k}\right)$, we properly smooth out H^S so that $H^S \in C^{n-1,\,\beta}(\mathbb{R}^n)$ (need not be rotationally symmetric; see § 3 g and § 3 h).

Main Theorem 1.7. *For $n \ge 6$, let $H^S \in C^{n-1,\,\beta}(\mathbb{R}^n)$ be as described in* (1.6), *with the parameter τ and η satisfying*

$$\tau \in (n-1\,,\ n)\,, \qquad 1 - A^2 \ > \ \eta \ > \ B^2 \ > \ 0 \quad \textit{and} \quad \frac{1+\eta}{1-\eta} \ \le \ \frac{5}{2}\,. \tag{1.8}$$

Here A and B are positive numbers. There exist positive constants C, C_1, c and ε_o so that if the parameters $\mathbf{a}$ and σ satisfy

$$\mathbf{a} \ > \ C^2 \quad \textit{and} \quad 0 \ < \ \sigma \ < \ c^2\,,$$

then the equation

$$\Delta_o\, v + \tilde{c}_n \left[\, 4n\,(n-1) + \varepsilon\, H^S \,\right] v^{\frac{n+2}{n-2}} \ = \ 0 \qquad \textit{in} \quad \mathbb{R}^n \tag{1.9}$$

has an infinite number of positive solutions $\{v_m\}_{m=1}^\infty \subset C^{2,\,\bar\beta}(\mathbb{R}^n)$ whenever $|\varepsilon| \le \varepsilon_o$. Moreover,

$$\|\, v_m - V_{\lambda_m\,,\ \xi_m} \|_{\triangledown} \ \le \ C_1 \cdot \varepsilon \qquad \textit{for} \quad m = 1,\ 2, \cdots. \tag{1.10}$$

Here $\lambda_m \ \to \ 0$ and $|\,\xi_m| \ \to \ 0$ as $m \ \to \ \infty$. As a result, 0 is a blow-up point for $\{v_m\}_{m=1}^\infty$.

In (1.10), $\|\ \ \|_\triangledown$ represents the L^2-norm on gradient for the Hilbert space $\mathcal{D}^{1,\,2}$. See (2.7) in Part I [16]. Via this L^2-norm on gradient and the Sobolev inequality, we discuss in § 3 j how to transfer these solutions $\{v_m\}_{m=1}^\infty$ back to S^n as a blow-up sequence of solutions for equation (1.1). [As for the number 5/2 in (1.8), it appears naturally in some calculations. It is, however, not absolute or sharp.]

§ 1 b. *Main features concerning the proof.* The key is to show that the critical point with respect to a single annular domain is 'stable' enough to withstand the interaction from other annular domains. Thus we are required to take $\mathbf{a}$ to be large (big spacing between the annular domains $\Longrightarrow$ less interference), and η to be not too small (producing stronger effect on the gradient change for an individual annular domain).

In § 2, we estimate the first derivatives of $G_{|\mathbf{z}}$ via the stereographic projection back to S^n (see § 2 a and § 2 b). This geometric approach helps to visualize the location of the critical point (cf. Lemma 2.21), and the sharp changes in gradient after leaving the critical point (cf. Lemma 2.65).

We remark that the method is stable under "small perturbation" so that functions nearby H^S can also be used in the construction. See § 3 e for more detail.

§ 1 c. *Conventions.* Throughout this work, we assume the dimension $n \ge 3$, except when otherwise is specifically mentioned, and let $\tilde{c}_n = (n-2)/[\,4\,(n-1)]$. We observe the practice of using C, possibly with sub-indices, to denote various positive constants, which may be rendered *differently* from line to line according to the contents. *Whilst we use $\bar{c}$ and $\bar{C}$, possibly with sub-index, to denote a* fixed *positive constant which always keeps the same value as it is first defined*. [The *negative* constant $\bar{c}_{-1}$ is defined in the sentence proceeding (1.5).]

$\bullet_1$ Denote by $B_y(r)$ the open ball in $\mathbb{R}^n$ with center at y and radius $r > 0$, and $\|\, S^n \,\|$ the measure of S^n in $\mathbb{R}^{n+1}$ with respect to the standard metric.

$\bullet_2$ Δ_g is the Laplace-Beltrami operator associated with the metric g on S^n. Likewise, Δ_o is the Laplace-Beltrami operator associated with the Euclidean metric g_o on $\mathbb{R}^n$, and Δ_1 is the Laplace-Beltrami operator associated with the standard metric g_1 on S^n.

$\bullet_3$ Whereever there is no risk of misunderstanding, we suppress dy from the integral expression.

§ 2. Annular domains.

Let us start with a general *smooth* and bounded domain $\Omega \subset \mathbb{R}^n$, and let H_Ω be the characteristic function of Ω (i.e., H equals to 1 within Ω, and 0 without). Expression (1.3) becomes

$$G_{|\mathbf{z}}\,(\lambda\,,\,\xi\,;\,\Omega) \;:=\; \bar{c}_{-1} \int_\Omega \left[\frac{\lambda}{\lambda^2 + |\,y-\xi|^{\,2}}\right]^n \qquad (H \;=\; H_\Omega)\,. \tag{2.1}$$

We remark that $G_{|\mathbf{z}}\,(\bullet\,,\,\bullet\,;\,\Omega)$ remains a smooth function on $\mathbb{R}^+ \times \mathbb{R}^n$.

§ 2 a. *Rescaling and shifting.* Consider the change of variables $\bar{y} = \lambda^{-1}\,(y-\xi)$. We have

$$\int_\Omega \left(\frac{\lambda}{\lambda^2 + |\,y-\xi|^{\,2}}\right)^n \;=\; \int_{\left(\frac{\Omega_{-\xi}}{\lambda}\right)} \left(\frac{1}{1+|\bar{y}|^{\,2}}\right)^n d\bar{y}\,. \tag{2.2}$$

Here we use a kind of "*geometric notations*" :

$$\Omega_{-\,\xi} \;:=\; \{\,\tilde{y} \in \mathbb{R}^n \;\;|\;\; \tilde{y} \;=\; y-\xi \quad \text{for} \;\; y \in \Omega\,\}\,, \tag{2.3}$$

$$\left(\frac{\Omega_{-\xi}}{\lambda}\right) := \{\,\bar{y} \in \mathbb{R}^n \mid \bar{y} = \lambda^{-1}\,\tilde{y} \quad \text{for} \;\; \tilde{y} \in \Omega_{-\,\xi}\,\} = \left\{\bar{y} \in \mathbb{R}^n \;|\;\; \bar{y} = \frac{y}{\lambda} - \frac{\xi}{\lambda} \quad \text{for} \;\; y \in \Omega\right\}.$$

§ 2 b. *Projection of annular regions.* Let $\dot{\mathcal{P}}$ be the stereographic projection which sends the north pole $\mathbf{N}$ to ∞. The conformal factor is given by $g_1\,(x) = \left[\dfrac{4}{(1+r^2)^{\,2}}\right] g_o\,(y)$ for $y \;=\; \dot{\mathcal{P}}(x)$. See, for instance, (2.1) and (2.3) in Part I [16].

We obtain

$$(2.4) \qquad \int_{\left(\frac{\Omega_{-\xi}}{\lambda}\right)} \left(\frac{1}{1+|\bar{y}|^2}\right)^n d\bar{y} \;=\; \frac{1}{2^n}\int_{\dot{\mathcal{P}}^{-1}\left(\frac{\Omega_{-\xi}}{\lambda}\right)} dS\,,$$

where dS refers to the standard measure on S^n. Suppose $\overline{B_o(R_c)} \subset B_o(R_b)$, where R_c and R_b are positive numbers. Consider the annular domain

$$\Omega \;=\; B_o\,(R_b) \setminus \overline{B_o(R_c)} \quad \Longrightarrow \quad \frac{\Omega_{-\,\xi}}{\lambda} \;=\; B_{\frac{-\xi}{\lambda}}\left(\lambda^{-1}R_b\right) \setminus \overline{B_{\frac{-\xi}{\lambda}}\,(\lambda^{-1}R_c)}\;.$$

Here we use (2.3). We are interested in finding

$$(2.5) \qquad \int_{\dot{\mathcal{P}}^{-1}\left(B_{\frac{-\xi}{\lambda}}\,(\lambda^{-1}\,R_b)\,\setminus\,\overline{B_{\frac{-\xi}{\lambda}}\,(\lambda^{-1}\,R_c\,)}\right)} dS\,.$$

It is remarked that the above expression depends smoothly on λ and ξ.

The stereographic projection $\dot{\mathcal{P}}$ sends an $(n-1)-$sphere on $S^n \setminus \{\mathbf{N}\}$ to an $(n-1)-$sphere on $\mathbb{R}^n$, and vis versa. See for example [11], pp. 7. It follows that the domain in the integral (2.5) can be rendered as taking away two 'caps' from S^n.

§ 2 c. *Volume of the cap in upper or lower hemisphere.* Consider a cap $\mathcal{C} \subset S^n$, bounded by the $(n-1)$-sphere with radius ϱ, **measured in the metric of** $\mathbb{R}^{n+1}$. *Here we only consider such a cap that can be contained in a hemisphere.* It follows that $\varrho < 1$. By symmetry, we view the cap as upon the southern hemisphere (resp. northern hemisphere), "centered" at the respective pole:

$$\text{Volume of the cap upon lower hemisphere} \;=\; \|S^{n-1}\| \cdot \int_0^{\arcsin\varrho} [\sin\dot{\phi}\,]^{n-1}\,d\,\dot{\phi}\,,$$

$$\text{Volume of the cap upon upper hemisphere} \;=\; \|S^{n-1}\| \cdot \int_0^{\arcsin\varrho} [\sin\phi\,]^{n-1}\,d\,\phi\,.$$

Here $\dot{\phi}$ is measured from the negative direction of the $x_{n+1}-$axis, and ϕ from the positive direction of $x_{n+1}-$axis. We write the last expression as

$$(2.6) \qquad \text{Vol Cap}\,(\varrho) \;=\; \|\,S^{n-1}\| \cdot \int_0^{\arcsin\,\varrho} [\sin\phi]^{n-1}\,d\,\phi\,,$$

where we understand that $\varrho < 1$, and possibly after a rotation, the cap is on the upper hemisphere. We can rewrite (2.5) as

$$(2.7) \qquad \int_{\mathcal{P}^{-1}\left(B_{\frac{-\xi}{\lambda}}(\lambda^{-1}R_b)\,\setminus\,\overline{B_{\frac{-\xi}{\lambda}}(\lambda^{-1}R_c\,)}\,\right)} dS \;=\; \|\,S^n\| \;-\; \text{Vol Cap}\,(\varrho_b) \;-\; \text{Vol Cap}\,(\varrho_c)\,,$$

where ϱ_b (respectively ϱ_c) is the radius of the sphere $\mathcal{P}^{-1}\left(\partial B_{\frac{-\xi}{\lambda}}\,(\,\lambda^{-1}R_b)\right)$ $\Big[$respectively $\mathcal{P}^{-1}\left(\partial B_{\frac{-\xi}{\lambda}}\,(\lambda^{-1}R_c\,)\right)\Big]$ (measured in the Euclidean metric of $\mathbb{R}^{n+1}$).

For later application, let us consider ϱ depending smoothly on a parameter δ, i.e., $\varrho = \varrho(\delta) < 1$. We then have

$$(2.8)\qquad \frac{d\,[\,\text{Vol Cap}\,(\varrho)]}{d\,\delta} = \|\,S^{n-1}\| \cdot [\,\sin\phi\,]^{\,n-1}\Big|_{\phi = \arcsin\,\varrho} \times \frac{1}{\sqrt{1-\varrho^2}} \cdot \frac{d\,\varrho(\delta)}{d\,\delta}$$

$$= \|\,S^{n-1}\| \cdot \frac{\varrho^{\,n-1}}{\sqrt{1-\varrho^2}} \cdot \frac{d\,\varrho(\delta)}{d\,\delta} = \|\,S^{n-1}\| \cdot \frac{\varrho^{\,n-1}}{\sqrt{1-\varrho^2}} \cdot \frac{1}{2\,\varrho}\,\frac{d\,[\,\varrho(\delta)]^2}{d\,\delta}$$

$$= \frac{\|\,S^{n-1}\|}{2} \cdot \frac{\varrho^{\,n-2}}{\sqrt{1-\varrho^2}} \cdot \frac{d\,[\,\varrho\,(\delta)]^2}{d\,\delta} \qquad (\,\varrho < 1)\,.$$

Note that the term

$$(2.9)\qquad \frac{\varrho^{\,n-2}}{\sqrt{1-\varrho^2}} \quad \text{increases as } \varrho \text{ increases} \qquad [\,\text{here } n \geq 3 \text{ and } \varrho \in [\,0,\ 1)\,]\,.$$

For example, when changing λ in (2.7) while fixing ξ $[\,\varrho_b = \varrho_b\,(\lambda) < 1$ and $\varrho_c = \varrho_c\,(\lambda) < 1\,]$, at the same time maintaining the cap $S^n \ \setminus\ \mathcal{P}^{-1}\left(B_{\frac{-\xi}{\lambda}}\,(\,\lambda^{-1}R_b)\right)$ to be on the northern hemisphere, and the cap $\mathcal{P}^{-1}\left(B_{\frac{-\xi}{\lambda}}\,(\lambda^{-1}R_c\,)\right)$ on the southern hemisphere, we then have

$$(2.10)\qquad \frac{\partial}{\partial\lambda}\int_{\mathcal{P}^{-1}\left(B_{\frac{-\xi}{\lambda}}\,(\,\lambda^{-1}R_b)\ \setminus\ \overline{B_{\frac{-\xi}{\lambda}}\,(\lambda^{-1}R_c\,)}\ \right)} dS$$

$$= \frac{\|\,S^{n-1}\|}{2} \cdot \left\{ \frac{\varrho_b^{n-2}}{\sqrt{1-\varrho_b^2}} \cdot \left[-\frac{d\,[\,\varrho_b\,(\lambda)]^2}{d\lambda} \right] + \frac{\varrho_c^{n-2}}{\sqrt{1-\varrho_c^2}} \cdot \left[-\frac{d\,[\,\varrho_c\,(\lambda)]^2}{d\lambda} \right] \right\}.$$

Bear in mind that the bigger the upper cap, the smaller the value in (2.5). Likewise, the bigger the lower cap, the smaller the value in (2.5). Similar expression can be found on derivatives in (the components of) ξ.

§ 2 d. *Symmetric case and the geometric mean.* The geometric expressions (2.5) and (2.6) provide light on the location of the critical point: the boundary spheres should have the same radius (when seen in $\mathbb{R}^{n+1}$) so as to cancel the increasing and decreasing effect (when λ is changed). In the following, we provide more details on this thought.

In (2.5), take

$$(2.11)\qquad R_b = t + \Delta\,, \qquad R_c = t - \Delta\,, \qquad \text{where } t > \Delta > 0 \text{ are given}\,.$$

Observe that t equals the *arithmetic mean* of the outer and inner radius. We can start with (2.1). Putting $\xi = 0$, we obtain

$$(2.12)\qquad \int_{B_o\,(t+\Delta)\setminus\overline{B_o\,(t-\Delta)}} \left(\frac{\lambda}{\lambda^2 + |\,y|^{\,2}}\right)^n = \|\,S^{n-1}\| \cdot \int_{t-\Delta}^{t+\Delta} \left(\frac{\lambda}{\lambda^2 + r^2}\right)^n r^{\,n-1}\,dr$$

$$= \|\,S^{n-1}\| \cdot \int_{\arctan\left(\frac{t-\Delta}{\lambda}\right)}^{\arctan\left(\frac{t+\Delta}{\lambda}\right)} \frac{[\,\tan\theta\,]^{\,n-1} \cdot [\,\sec\theta]^2}{[\sec\theta]^{2n}}\,d\theta \qquad (\,r = \lambda\tan\theta\,)$$

$$= \| S^{n-1} \| \int_{\arctan\left(\frac{t-\Delta}{\lambda}\right)}^{\arctan\left(\frac{t+\Delta}{\lambda}\right)} [\sin\theta\,\cos\theta]^{\,n-1}\, d\theta = \frac{\| S^{n-1} \|}{2^n} \int_{\arctan\left(\frac{t-\Delta}{\lambda}\right)}^{\arctan\left(\frac{t+\Delta}{\lambda}\right)} [\sin(2\theta)]^{\,n-1}\; d\,(2\theta)$$

$$= \frac{\| S^{n-1} \|}{2^n} \int_{2\arctan\left(\frac{t-\Delta}{\lambda}\right)}^{2\arctan\left(\frac{t+\Delta}{\lambda}\right)} [\sin\varphi]^{\,n-1}\; d\varphi \qquad (\,\varphi = 2\theta)\,.$$

$$(2.13) \qquad \cdots\cdots \implies \frac{\partial}{\partial\lambda} \int_{2\arctan\left(\frac{t-\Delta}{\lambda}\right)}^{2\arctan\left(\frac{t+\Delta}{\lambda}\right)} [\sin\varphi]^{\,n-1}\; d\varphi$$

$$= -\frac{2\,[\,t+\Delta]}{[\,t+\Delta]^2+\lambda^2}\,(\,\sin\overline{\varphi}_+)^{\,n-1} + \frac{2\,[\,t-\Delta]}{[\,t-\Delta]^2+\lambda^2}\,(\,\sin\overline{\varphi}_-)^{\,n-1}\,,$$

where $\qquad \overline{\varphi}_+ = 2\arctan\left(\dfrac{t+\Delta}{\lambda}\right)\,, \qquad \overline{\varphi}_- = 2\arctan\left(\dfrac{t-\Delta}{\lambda}\right)\,.$

Let us take

$$(2.14) \qquad \lambda^2 \;=\; \lambda_M^2 \;:=\; t^2 \;-\; \Delta^2 \qquad \implies \qquad \lambda_M \;=\; \sqrt{(t\,+\,\Delta)(t\,-\,\Delta)}\;\;.$$

That is, λ_M is the *geometric mean* of the inner and outer radius. Consider (2.13) :

$$(2.15) \qquad \frac{2\,[\,t\,+\,\Delta]}{[\,t\,+\,\Delta]^2+\lambda^2} \;=\; \frac{2}{(t\,+\,\Delta)\,+\,(t\,-\,\Delta)} \;=\; \frac{1}{t}\,,$$

$$(2.16) \qquad \frac{2\,[\,t\,-\,\Delta]}{[\,t\,-\,\Delta]^2+\lambda_M^2} \;=\; \frac{2}{(t\,-\,\Delta)\,+\,(t\,+\,\Delta)} \;=\; \frac{1}{t}\,,$$

$$(2.17) \qquad \bar{\varphi}_{M_-} \;=\; 2\arctan\left(\frac{t\,-\,\Delta}{\lambda_M}\right) \;=\; 2\arctan\sqrt{\frac{t\,-\,\Delta}{t\,+\,\Delta}}\;,$$

$$(2.18) \qquad \bar{\varphi}_{M_+} \;=\; 2\arctan\left(\frac{t\,+\,\Delta}{\lambda_M}\right) \;=\; 2\arctan\sqrt{\frac{t\,+\,\Delta}{t\,-\,\Delta}}\;.$$

We note that

$$\theta_- \;=\; \arctan\chi \qquad \iff \qquad \tan\,\theta_- \;=\; \chi\,,$$

$$\theta_+ \;=\; \arctan\frac{1}{\chi} \qquad \iff \qquad \tan\,\theta_+ \;=\; \frac{1}{\chi} \qquad\qquad (\,\text{here} \quad 0 \,<\, \theta_-\,, \quad \theta_+ \,<\, \pi/2\,)$$

$$\implies [\tan\,\theta_-]\cdot[\tan\,\theta_+] \;=\; 1 \quad \implies \quad \frac{[\sin\,\theta_-]\cdot[\sin\,\theta_+]}{[\cos\,\theta_-]\cdot[\cos\,\theta_+]} - 1 \;=\; 0$$

$$\implies \frac{[\sin\,\theta_-]\cdot[\sin\,\theta_+] - [\cos\,\theta_-]\cdot[\cos\,\theta_+]}{[\cos\,\theta_-]\cdot[\cos\,\theta_+]} \;=\; 0 \quad \implies \quad \cos\,(\,\bar{\varphi}_{M_+} + \bar{\varphi}_{M_-}\,) \;=\; 0$$

$$\implies \theta_+ + \theta_- = \frac{\pi}{2} \quad \implies \quad \theta_+ \;=\; \frac{\pi}{2} - \theta_- \qquad\qquad (\text{as} \;\; 0 \,<\, \theta_-\,,\;\; \theta_+ \,<\, \pi/2\,)$$

$$\implies \bar{\varphi}_{M_+} \;=\; \pi - \bar{\varphi}_{M_-} \qquad (\implies \;\; \text{domain} \;\; \text{symmetric}) \quad \implies \; \sin\,\bar{\varphi}_{M_+} \;=\; \sin\,\bar{\varphi}_{M_-}$$

$$(2.19)\,\cdots \implies \;\; (\,\sin\,\bar{\varphi}_{M_+})^{\,n-1} \;=\; (\,\sin\,\bar{\varphi}_{M_-})^{\,n-1}.$$

Together with (2.13), (2.15)– (2.19), we arrive at

$$\left[\frac{\partial}{\partial\lambda} \int_{2\arctan\left(\frac{t-\Delta}{\lambda}\right)}^{2\arctan\left(\frac{t+\Delta}{\lambda}\right)} [\sin\varphi]^{n-1}\, d\varphi \right]_{\lambda=\lambda_M} = 0\,. \tag{2.20}$$

Lemma 2.21. *Given any positive numbers* t *and* Δ *with* $t > \Delta$, *let* H *be equal to one in the annular domain* $\Omega = B_o(t+\Delta) \setminus \overline{B_o(t-\Delta)}$, *and zero outside, and* λ_M *given by*

$$\lambda_M = \sqrt{(t+\Delta)(t-\Delta)}\,. \tag{2.22}$$

Then $(\lambda_M\,,\ \vec{0}\,)$ *is a critical point for* $G_{|\mathbf{z}}(\bullet\,,\,\bullet\,;\,\Omega)$ *given in* (2.1).

Proof. As $G_{|\mathbf{z}}(\bullet\,,\,\bullet\,;\,\Omega)$ is differentiable, we first let $\xi = 0$ in the expression (2.1), and apply the calculations in (2.11) – (2.20) to conclude that $\left.\frac{\partial G_{|\mathbf{z}}}{\partial\lambda}\right|_{(\lambda_M\,,\ \vec{0}\,)} = 0$. In light of Lemma 6.4 in Part I [16], we also have $\left.\frac{\partial G_{|\mathbf{z}}}{\partial\xi_j}\right|_{(\lambda_M\,,\ \vec{0}\,)} = 0$ for $j = 1,\ 2, \cdots,\ n$. □

With a closer look, we find that λ_M is the only critical point [for the expressions in (2.12)]. As this point is not used in this article, we direct the interested readers to see § A .12 in the e-Appendix. Next, we show that it is a non-degenerate critical point.

Lemma 2.23. *Under the notations and the conditions in Lemma* 2.21, $(\lambda_M\,,\ \vec{0}\,)$ *is a non-degenerate critical point for* $G_{|\mathbf{z}}(\bullet\,,\,\bullet\,;\,\Omega)$.

Proof. We assert that

$$\frac{\partial^2 G_{|\mathbf{z}}}{\partial\lambda^2}(\lambda_M\,,\ \vec{0}\,) < 0\,. \tag{2.24}$$

In view of Lemma 7.7 and Proposition 7.10 in Part I [16], and the symmetry:

$$\frac{\partial^2 G_{|\mathbf{z}}}{\partial\,\xi_\ell^2}(\lambda_M\,,\ \vec{0}\,) = \frac{\partial^2 G_{|\mathbf{z}}}{\partial\,\xi_j^2}(\lambda_M\,,\ \vec{0}\,) \qquad \text{for} \qquad 1 \le j\,,\ \ell \le n\,,$$

(2.24) is enough to complete the proof. To do the task, we continue from (2.13),

$$\left.\frac{\partial^2}{\partial\lambda^2} \int_{2\arctan\left(\frac{t-\Delta}{\lambda}\right)}^{2\arctan\left(\frac{t+\Delta}{\lambda}\right)} [\sin\varphi]^{n-1}\, d\varphi\,\right|_{\lambda=\lambda_M}$$

$$= \frac{4\,[\,t+\Delta]\,\lambda_M}{\{[\,t+\Delta]^2+\lambda_M^2\}^2}\,[\sin\bar\varphi_{M_+}]^{n-1} + \frac{4\,(t+\Delta)^2}{\{[\,t+\Delta]^2+\lambda_M^2\}^2}\,(n-1)\,[\sin\bar\varphi_{M_+}]^{n-2}\cos\bar\varphi_{M_+}$$

$$- \frac{4\,[\,t-\Delta]\,\lambda_M}{\{[\,t-\Delta]^2+\lambda_M^2\}^2}\,[\sin\bar\varphi_{M_-}]^{n-1} - \frac{4\,(t-\Delta)^2}{\{[\,t-\Delta]^2+\lambda_M^2\}^2}\,(n-1)\,[\sin\bar\varphi_{-}]^{n-2}\cos\bar\varphi_{M_-}$$

$$= \frac{1}{t^2}\left(\frac{\lambda_M}{t+\Delta}\right)[\sin\bar{\varphi}_{M_+}]^{n-1} \;-\; \frac{1}{t^2}\left(\frac{\lambda_M}{t-\Delta}\right)[\sin\bar{\varphi}_{M_-}]^{n-1}$$

[$\uparrow$ the difference is negative, via (2.17 – (2.19)]

$$+\ \frac{n-1}{t^2}\cdot[\sin\,\varphi_{M_\pm}]^{n-2}\times[\cos\bar{\varphi}_{M_+}-\cos\bar{\varphi}_{M_-}] \;<\; 0\,.$$

Here we use the notations introduced in (2.13), and (2.20). In the last step above, we make use of (2.17) and (2.18) to obtain

$$\pi \;>\; \bar{\varphi}_{M_+} \;>\; \frac{\pi}{2} \quad\Longrightarrow\quad \cos\bar{\varphi}_{M_+}<0\,, \qquad \frac{\pi}{2}>\bar{\varphi}_{M_-} \;>\; 0 \quad\Longrightarrow\quad \cos\bar{\varphi}_{M_-} \;>\; 0\,.$$

With this, we arrive to (2.24). □

It follows from (2.24), Lemma 7.7 and Proposition 7.10 in Part I [16], that

$$\text{(2.25)}\qquad \mathrm{Deg}\,(\,\nabla G_{|\mathbf{z}}\,,\ B_{\mathbf{p_M}}(r)\,,\ \vec{0}\,) \;=\; -1\,, \qquad \text{where}\quad \mathbf{p_M} \;:=\; (\lambda_M\,,\ \vec{0})\,,$$

provided that $\nabla G_{|\mathbf{z}}(\lambda\,,\ \xi)\neq\vec{0}$ for all $(\lambda\,,\ \xi)\in\overline{B_{\mathbf{p_M}}(r)}\setminus\{\mathbf{p_M}\}$. See [9] and also § A.3 in the e-Appendix.

In the process on superimposing annular domains, we must show that effects from other annular domains do not affect the stability of the one we focus on. For this, we require more information on the strength of the first derivative [at points of the boundary of $B_{\mathbf{p_M}}(r)$]. The following result is a prelude to this (cf. § 2 h).

Lemma 2.26. *Under the notations and the conditions in Lemma* 2.21, *assume that*

$$\text{(2.27)}\qquad \sqrt{\frac{5}{2}} \;\ge\; \frac{t\,+\,\Delta}{\lambda_M} \;\ge\; \frac{t\,-\,\Delta}{\lambda_M} \;\ge\; \sqrt{\frac{2}{5}}\,.$$

There exist positive constant $\bar{\varepsilon}_6$ *and* $\bar{\varepsilon}_7$ *such that*

$$\text{(2.28)}\qquad \left|\frac{\partial\,G_{|\mathbf{z}}}{\partial\lambda}(\lambda_M+s,\ \vec{0})\right| \;>\; [\,\bar{C}_6\cdot\bar{\varepsilon}_6\,]\cdot\frac{1}{\lambda_M} \qquad \textit{for}\quad \bar{\varepsilon}_7 \;\ge\; \frac{|\,s|}{\lambda_M} \;\ge\; \bar{\varepsilon}_6\,.$$

The numbers $\bar{C}_6$, $\bar{\varepsilon}_6$ *and* $\bar{\varepsilon}_7$ *can be chosen to be independent on* t *and* Δ, *as long as* (2.27) *is fulfilled.*

Proof. From (2.14) [cf. (2.26)], we obtain

$$\text{(2.29)}\qquad \frac{\partial^2}{\partial\lambda^2}\int_{2\arctan\left(\frac{t-\Delta}{\lambda}\right)}^{2\arctan\left(\frac{t+\Delta}{\lambda}\right)}[\sin\varphi]^{n-1}\,d\varphi\,\Big|_{\lambda=\lambda_M+s}$$

$$= \frac{4\,[\,t+\Delta]\,[\,\lambda_M+s]}{\{[\,t+\Delta]^2+[\,\lambda_M+s]^2\}^2}\cdot[\sin\bar{\varphi}_+]^{n-1} + \frac{4\,(n-1)\,(t+\Delta)^2}{\{[\,t+\Delta]^2+[\,\lambda_M+s]^2\}^2}\cdot[\sin\bar{\varphi}_+]^{n-2}\cos\bar{\varphi}_+$$

$$-\frac{4\,[\,t-\Delta]\,[\,\lambda_M+s]}{\{[\,t-\Delta]^2+[\,\lambda_M+s]^2\}^2}\cdot[\sin\bar{\varphi}_-]^{n-1} - \frac{4\,(n-1)\,(t-\Delta)^2}{\{[\,t-\Delta]^2+[\,\lambda_M+s]^2\}^2}\cdot[\sin\bar{\varphi}_-]^{n-2}\cos\bar{\varphi}_-\,.$$

As in (2.13),

$$\begin{aligned}
\bar{\varphi}_- \ &= \ 2\arctan\left(\frac{t\,-\,\Delta}{\lambda_M + s}\right) \ = \ 2\arctan\left(\frac{t\,-\,\Delta}{\lambda_M}\cdot\left[1\,+\,O\left(\frac{s}{\lambda_M}\right)\right]\right) \\
&= \ 2\arctan\left(\frac{t\,-\,\Delta}{\lambda_M}\,+\,O\left(\frac{s}{\lambda_M}\right)\right) \qquad [\text{via } \ (2.27)] \\
&= \ 2\arctan\left(\frac{t\,-\,\Delta}{\lambda_M}\right)\,+\,O\left(\frac{s}{\lambda_M}\right) \qquad (\text{first order approximation of arctan}) \\
&= \ \bar{\varphi}_{M_-}\,+\,O\left(\frac{s}{\lambda_M}\right) \qquad \text{for } \ \lambda_M^{-1}\cdot|\,s| \ \text{ small } \ [\text{cf. } (2.17)]\,.
\end{aligned}$$

Here we use the expansion

$$\frac{1}{\lambda_M + s} \ = \ \frac{1}{\lambda_M}\left(\frac{1}{1+\lambda_M^{-1}\,s}\right) \ = \ \frac{1}{\lambda_M}\left[\,1\,+\,O\,\left(\lambda_M^{-1}\,s\right)\right] \qquad \text{for } \ \lambda_M^{-1}\cdot|\,s| \ \text{ small}\,,$$

and the bound on the derivative of arctan. Similarly, $\bar{\varphi}_+ = \bar{\varphi}_{M_+} + O\left(\frac{s}{\lambda_M}\right)$ for $\lambda_M^{-1}\cdot|\,s|$ small, where we use (2.27) again. Moreover,

$$\begin{aligned}
&\frac{4\,[\,t\,+\,\Delta]\,[\,\lambda_M\,+\,s]}{\{[\,t\,+\,\Delta]^2+[\,\lambda_M\,+\,s]^2\}^2} \ = \ \frac{1}{\lambda_M^2}\cdot\frac{4\,[(\lambda_M^{-1}\,(t\,+\,\Delta)]\,[\,1\,+\,\lambda_M^{-1}\,s]}{\{[\,\lambda_M^{-1}\,(t\,+\,\Delta)]^2\,+\,[\,1\,+\,\lambda_M^{-1}\,s]^2\}^2} \\
&= \ \frac{1}{\lambda_M^2}\cdot\left[\frac{4\,[(\lambda_M^{-1}\,(t\,+\,\Delta)]}{\{[\,\lambda_M^{-1}\,(t\,+\,\Delta)]^2\,+\,1\}^2}\,+\,O\left(\frac{s}{\lambda_M}\right)\right] = \frac{1}{t^2}\left(\frac{\lambda_M}{t\,+\,\Delta}\right)\,+\,\frac{1}{\lambda_M^2}\cdot O\left(\frac{s}{\lambda_M}\right) \\
&= \ \frac{1}{t^2}\left(\frac{\lambda_M}{t\,+\,\Delta}\right)+\frac{1}{t^2}\cdot O\left(\frac{s}{\lambda_M}\right)\,, \qquad \text{as} \quad (2.27) \quad \Longrightarrow \ \sqrt{\frac{5}{2}} \ \ge \ \frac{t}{\lambda_M} \ \ge \ \sqrt{\frac{2}{5}}\,.
\end{aligned}$$

We apply a similar method to estimate the other terms in (2.29), and obtain

(2.30)
$$\frac{\partial^2}{\partial\lambda^2}\int_{2\arctan\left(\frac{t-\Delta}{\lambda}\right)}^{2\arctan\left(\frac{t+\Delta}{\lambda}\right)}\,[\sin\varphi]^{\,n-1}\ d\varphi\,\Bigg|_{\lambda\,=\,\lambda_M\,+\,s}$$

$$= \ \underbrace{\left[\ \frac{1}{t^2}\left(\frac{\lambda_M}{t\,+\,\Delta}\right)[\sin\bar{\varphi}_{M_+}]^{\,n-1}\,-\,\frac{1}{t^2}\left(\frac{\lambda_M}{t\,-\,\Delta}\right)[\sin\bar{\varphi}_{M_-}]^{\,n-1}\ \right]}_{[\text{as in the proof of (2.24)} \ \uparrow \ \text{the difference is negative}]} \ +$$

$$+\,\frac{n-1}{t^2}\cdot[\,\sin\,\varphi_{M_\pm}]^{\,n-2}\cdot[\,\cos\bar{\varphi}_{M_+}\,-\,\cos\bar{\varphi}_{M_-}]\,+\,\frac{1}{t^2}\cdot O\left(\frac{s}{\lambda_M}\right)\,.$$

We make use of (2.17), (2.18), (2.27) to deduce that there are positive constants c_1, c_2, c_3 and c_4 [independent on t and Δ as long as (2.27) is fulfilled] so that

$$\begin{aligned}
\pi - c_1^2 \ > \ \bar{\varphi}_+ \ > \ \frac{\pi}{2}+c_1^2 \quad &\Longrightarrow \quad \cos\bar{\varphi}_+ \ < \ -c_2^2 \quad \text{and} \quad \sin\varphi_{M_+} \ge c_2^2 \ > \ 0\,, \\
\frac{\pi}{2} - c_3^2 \ > \ \bar{\varphi}_- \ > \ c_3^2 \ > \ 0 \quad &\Longrightarrow \quad \cos\bar{\varphi}_- \ > \ c_4^2 \quad \text{and} \quad \sin\varphi_{M_-} \ \ge \ c_4^2 \ > \ 0\,.
\end{aligned}$$

Hence we can find a positive number $\bar{\varepsilon}_7$ so that

$$\frac{\partial^2}{\partial \lambda^2} \int_{2\arctan\left(\frac{t-\Delta}{\lambda}\right)}^{2\arctan\left(\frac{t+\Delta}{\lambda}\right)} [\sin\varphi]^{n-1}\, d\varphi \Bigg|_{\lambda=\lambda_M+s} \le -\frac{C_1}{t^2} + \frac{C_2}{t^2}\cdot\left(\frac{s}{\lambda_M}\right) \le -\frac{C_3}{t^2}$$

for $\lambda_M^{-1}\cdot|s| \le \bar{\varepsilon}_7$. Consider the case $s > 0$ first. Lemma 2.21 and an integration yield

$$\frac{\partial}{\partial \lambda} \int_{2\arctan\left(\frac{t-\Delta}{\lambda}\right)}^{2\arctan\left(\frac{t+\Delta}{\lambda}\right)} [\sin\varphi]^{n-1}\, d\varphi \Bigg|_{\lambda=\lambda_M+s} \le -\frac{C_3}{t^2}\cdot s \le -\frac{C_4}{\lambda_M}\cdot\frac{s}{\lambda_M} \le -\frac{C_4}{\lambda_M}\cdot\bar{\varepsilon}_6$$

for $\bar{\varepsilon}_6 \le \lambda_M^{-1}\cdot s \le \bar{\varepsilon}_7$. Here we use (2.27) again. That is, we have (2.28) for $s > 0$. Similarly, we can handle the case $s < 0$. $\square$

§ 2 f. *Perturbation in* ξ. In order to apply Theorem 4.26 in Part I [16], we desire to provide lower bounds on $|\bigtriangledown G_{|\mathbf{z}}(\lambda, \xi)|$ off the critical point $(\lambda_M\ \vec{0})$. This means that we need to vary ξ as well as λ. According to the geometric formula (2.5), $B_{\lambda^{-1}\xi}(\lambda^{-1}R)$ is the ball in $\mathbb{R}^n$ with center at $\lambda^{-1}\xi$ and radius $\lambda^{-1}R$. In accordance,

$$\dot{\mathcal{P}}^{-1}(\partial B_{\lambda^{-1}\xi}(\lambda^{-1}R)) \tag{2.31}$$

is a $(n-1)$-sphere in $S^n \subset \mathbb{R}^{n+1}$. [In (2.31), R is a positive number.] It is harder to visualize the effect when ξ is changed. To proceed, we find the (Euclidean) radius ϱ of this $(n-1)$-sphere.

Via a rotation, we assume that ξ is along the *positive* y_1-axis. Granted this, let

$$\xi = (\delta, 0, \cdots, 0), \tag{2.32}$$

where we consider small perturbation so that $0 \le \delta < R$. The equation of $\partial B_{\lambda^{-1}\xi}(\lambda^{-1}R)$ is

$$(y_1 - \delta_\lambda)^2 + y_2^2 + \cdots + y_n^2 = R_\lambda^2, \tag{2.33}$$

where

$$R_\lambda = \frac{R}{\lambda} \quad \text{and} \quad \delta_\lambda = \frac{\delta}{\lambda}. \tag{2.34}$$

(2.33) and (2.34) imply

$$r^2 - 2y_1\cdot\delta_\lambda = R_\lambda^2 - \delta_\lambda^2 \quad\Longrightarrow\quad \frac{1 + x_{n+1}}{1 - x_{n-1}} - \frac{2\,\delta_\lambda\cdot x_1}{1 - x_{n+1}} = R_\lambda^2 - \delta_\lambda^2.$$

(As usual, $r^2 = y_1^2 + \cdots + y_n^2$.) See (2.1) and (2.2). The result is

$$(-2\,\delta_\lambda)\,x_1 + x_{n+1}(1 + R_\lambda^2 - \delta_\lambda^2) = (R_\lambda^2 - \delta_\lambda^2 - 1). \tag{2.35}$$

That is,

$$\dot{\mathcal{P}}^{-1}(\partial B_{\lambda^{-1}\xi}(\lambda^{-1}R)) = S^n \cap \{\text{The hyperplane defined by equation (2.35)}\} \tag{2.36}$$

To find the Euclidean radius, we measure the distance in $\mathbb{R}^{n+1}$ of the inverse images of the end points

$$-(R_\lambda - \delta_\lambda) \qquad \text{and} \qquad (R_\lambda + \delta_\lambda)\,, \tag{2.37}$$

respectively (these two points lie on the y_1-axis). In the plane defined by x_1-axis and x_{n+1}-axis, the inverse images (via $\dot{\mathcal{P}}^{-1}$) of these two points have coordinates given by

$$\left(-\frac{2(R_\lambda - \delta_\lambda)}{(R_\lambda - \delta_\lambda)^2 + 1}\,,\ \frac{(R_\lambda - \delta_\lambda)^2 - 1}{(R - \delta_\lambda)^2 + 1}\right), \qquad \left(\frac{2(R_\lambda + \delta_\lambda)}{(R + \delta_\lambda)^2 + 1}\,,\ \frac{(R_\lambda + \delta_\lambda)^2 - 1}{(R_\lambda + \delta_\lambda)^2 + 1}\right),$$

respectively. Refer to (2.1) and (2.2). As $2\,\varrho$ is the distance between these two points (measured in the standard Euclidean metric in $\mathbb{R}^{n+1}$), a calculation shows that (refer to § A.13 in the e-Appendix)

$$(2\varrho)^2 = \frac{(4R_\lambda)^2\,[\,(1 + R_\lambda^2 - \delta_\lambda^2)^2 + 4\,\delta_\lambda^2\,]}{[(R_\lambda + \delta_\lambda)^2 + 1]^2\ [(R_\lambda - \delta_\lambda)^2 + 1]^2}\,. \tag{2.38}$$

Compare with the non-perturbed case [i.e., when $\xi = 0$] which can also be found directly by using (2.2) on the points R_λ and $-R_\lambda$ along the x_1-axis:

$$(2\varrho)^2 = \left[\frac{2R_\lambda}{1 + R_\lambda^2} - \frac{(-2R_\lambda)}{1 + R_\lambda^2}\right]^2 = \frac{(4R_\lambda)^2}{(1 + R_\lambda^2)^2}\ \left(= (4R)^2 \cdot \frac{\lambda^2}{(R^2 + \lambda^2)^2}\right) \quad \text{for } \xi = 0\,. \tag{2.39}$$

For later reference in § 2 h, we differential both sides of (2.39) and obtain the following.

$$\frac{\partial\,(2\varrho)^2}{\partial \lambda} = \frac{(2\lambda)\,(4R)^2}{(R^2 + \lambda^2)^3} \cdot (R^2 - \lambda^2) = \frac{1}{\lambda}\left[\frac{2 \cdot (4R_\lambda)^2}{(R_\lambda^2 + 1)^3}\,(R_\lambda^2 - 1)\right] \qquad \text{for } \xi = 0\,. \tag{2.40}$$

§ 2 g. *Derivative in* ξ. Recall that we assume $\xi = (\delta\,,\, 0\,,\, \cdots,\, 0)$. When differentiating the denominator in (2.38), we make use of the following.

$$\begin{aligned} & \frac{1}{[(R_\lambda + \delta_\lambda)^2 + 1\,] \cdot [(R_\lambda - \delta_\lambda)^2 + 1\,]} \\ = \ & \frac{1}{[\,(R_\lambda^2 + \delta_\lambda^2 + 1) + 2R_\lambda \cdot \delta_\lambda] \cdot [\,(R_\lambda^2 + \delta_\lambda^2 + 1) - 2R_\lambda \cdot \delta_\lambda\,]} \\ = \ & \frac{1}{(R_\lambda^2 + \delta_\lambda^2 + 1)^2 - 4R_\lambda^2 \cdot \delta_\lambda^2} \qquad (\text{cancelation of first order terms in } \delta_\lambda) \\ = \ & \frac{1}{(1 + R_\lambda^2\,)^2 + \delta_\lambda^2\,(2 + \delta_\lambda^2 - 2R_\lambda^2\,)} \qquad \left(\text{recall } \delta_\lambda = \frac{\delta}{\lambda}\right). \end{aligned} \tag{2.41}$$

One finds that

$$
\begin{aligned}
(2.42)\qquad \frac{\partial\,(2\varrho)^2}{\partial\delta} \;&=\; -\,\frac{\delta_\lambda}{\lambda}\cdot\frac{(4R_\lambda)^2}{[(R_\lambda+\delta_\lambda)^2\,+\,1\,]^2\,[(R_\lambda-\delta_\lambda)^2\,+\,1\,]^2}\;\times\\
&\times\left\{4\,[\,(R_\lambda^2-1)-\delta_\lambda^2\,]-\frac{8\,[(1+R_\lambda^2-\delta_\lambda^2)^2\,+\,4\,\delta_\lambda^2\,]}{[(R_\lambda+\delta_\lambda)^2\,+\,1\,]\,[(R_\lambda-\delta_\lambda)^2\,+\,1\,]}\cdot\Big[\,(R_\lambda^2-1)\;-\delta_\lambda^2\,\Big]\right\}\\
=\;&-\,\frac{\delta_\lambda}{\lambda}\cdot[\,(R_\lambda^2-1)-\delta_\lambda^2\,]\,\times\frac{(4R_\lambda)^2}{[(R_\lambda+\delta_\lambda)^2\,+\,1]^2\,[(R_\lambda-\delta_\lambda)^2\,+\,1\,]^2}\times\\
&\times\left\{4\;-\;\frac{8\,[(1\,+\,R_\lambda^2-\delta_\lambda^2)^2+4\,\delta_\lambda^2\,]}{[(R_\lambda+\delta_\lambda)^2\,+\,1\,]\,[(R_\lambda-\delta_\lambda)^2\,+\,1\,]}\right\}.
\end{aligned}
$$

Continuing from the last line in (2.41), one can approximate

$$
\begin{aligned}
(2.43)\qquad &\frac{1}{[(R_\lambda+\delta_\lambda)^2\,+\,1\,]\cdot[(R_\lambda-\delta_\lambda)^2\,+\,1\,]}\\
&=\;\frac{1}{(1+R_\lambda^2\,)^2}\cdot\frac{1}{1+\delta_\lambda^2\cdot\frac{(\,2\,+\,\delta_\lambda^2\,-\,2R_\lambda^2)}{(1\,+\,R_\lambda^2)^2}}\;=\;\frac{1}{(1+R_\lambda^2\,)^2}\;+\;O\,(\delta_\lambda^2)
\end{aligned}
$$

for $\delta_\lambda > 0$ small, and $R_\lambda \le \sqrt{\frac{5}{2}}$. It follows that

$$
\begin{aligned}
(2.44)\qquad \frac{\partial\,(2\varrho)^2}{\partial\delta}\;&=\;-\frac{\delta_\lambda}{\lambda}\cdot[\,(R_\lambda^2-1)-\delta_\lambda^2\,]\cdot\left[\frac{(4R_\lambda)^2}{(1+R_\lambda^2)^4}+O\,(\delta_\lambda^2)\right]\cdot[\,4-8+O\,(\delta_\lambda^2)]\\
&=\;\frac{\delta_\lambda}{\lambda}\cdot\left[(R_\lambda^2-1)\cdot\frac{4\cdot(4R_\lambda)^2}{(1+R_\lambda^2)^4}+O\,(\delta_\lambda^2)\right]
\end{aligned}
$$

for $\delta_\lambda > 0$ small, and $R_\lambda \le \sqrt{\frac{5}{2}}$. Formula (2.44) indicates that

$$
\frac{\partial\,(2\varrho)^2}{\partial\delta} > 0 \;\;\text{when}\;\; R_\lambda > 1\,; \qquad \text{whereas}\;\; \frac{\partial\,(2\varrho)^2}{\partial\delta} < 0 \;\;\text{when}\;\; R_\lambda < 1
$$

for δ_λ small enough (relative to R_λ) .

Expansion for small variation of λ_M *.* Take

$$
(2.45)\qquad \lambda=\lambda_M+s \;\;\text{and}\;\; R=t+\Delta
$$

$$
\Longrightarrow\quad R_\lambda^2=\left(\frac{t+\Delta}{\lambda}\right)^2=\left(\frac{t+\Delta}{\lambda_M+s}\right)^2=\left(\frac{t+\Delta}{\lambda_M}\right)^2+O\left(\frac{s}{\lambda_M}\right)=\underset{(\uparrow\ \text{cf.}\ \ (2.43))}{\frac{t+\Delta}{t-\Delta}}+O\left(\frac{s}{\lambda_M}\right)
$$

$$
\Longrightarrow(R_\lambda^2-1)\cdot\frac{4\cdot(4R_\lambda)^2}{(1+R_\lambda^2)^4}=\frac{\frac{128\Delta}{t-\Delta}\cdot\frac{t+\Delta}{t-\Delta}}{\left(\frac{2t}{t-\Delta}\right)^4}+O\left(\lambda_M^{-1}s\right)=\frac{128\Delta\,(t+\Delta)(t-\Delta)^2}{(2t)^4}+O\left(\lambda_M^{-1}s\right)
$$

$$
\Longrightarrow\qquad \left.\frac{\partial\,(2\varrho)^2}{\partial\delta}\right|_{R\,=\,t\,+\,\Delta}\;=\;\frac{\delta_\lambda}{\lambda}\cdot\left[4^3\cdot\frac{2\Delta\,(t+\Delta)(t-\Delta)^2}{(2t)^4}\;+\;O\left(\frac{s}{\lambda_M}\right)\;+\;O\,(\delta_\lambda^2)\right]
$$

for δ_λ and $\lambda_M^{-1}\,|\,s|$ small, and $R_\lambda \le \sqrt{\frac{5}{2}}$. Likewise, take $\lambda = \lambda_M + s$ and $R = t - \Delta$, we obtain

$$(2.46) \qquad \frac{\partial\,(2\varrho)^2}{\partial\delta}\Big|_{R\,=\,t\,-\,\Delta} \;=\; -\,\frac{\delta_\lambda}{\lambda}\cdot\left[4^3\cdot\frac{2\Delta\,(t-\Delta)(t+\Delta)^2}{(2t)^4} \;+\; O\left(\frac{s}{\lambda_M}\right) \;+\; O\,(\delta_\lambda^2)\right]$$

for δ_λ and $\lambda_M^{-1}\,|\,s|$ small, and $R_\lambda \le \sqrt{\frac{5}{2}}$. In the following discussion, we restrict ourselves to the case

$$(2.47) \qquad 1 \,-\, A^2 \;\ge\; \frac{\Delta}{t} \;\ge\; B^2 \;>\; 0$$

for some (fixed) positive numbers A and B, and

$$(2.48) \qquad \sqrt{\frac{5}{2}} \;\ge\; \frac{t\,+\,\Delta}{\lambda_M} \;>\; \frac{t\,-\,\Delta}{\lambda_M} \;\ge\; \sqrt{\frac{2}{5}}\;.$$

Lemma 2.49. *Under the notations and conditions in Lemma* 2.21, *assume also* (2.47) *and* (2.48). *Let*

$$\lambda \;=\; \lambda_M \,+\, s \quad \textit{and} \quad \delta \;=\; |\,\xi\,|\,.$$

There exist positive constants $\bar\varepsilon_8$, $\bar\varepsilon_9$, $\bar\varepsilon_{10}$ *and* $\bar C_7$ *such that if*

$$(2.50) \qquad \bar\varepsilon_8 \;\ge\; \lambda_M^{-1}\,\delta \;\ge\; \bar\varepsilon_9 \;>\; 0 \qquad \textit{and} \qquad \lambda_M^{-1}\,|\,s| \;\le\; \bar\varepsilon_{10}\,,$$

then we have

$$(2.51) \qquad \|\,\nabla_\xi\; G_{|\mathbf{z}}\,(\lambda_M + s\,,\;\xi)\| \;=\; \sqrt{\sum_{\ell=1}^n \left|\,\frac{\partial G_{|\mathbf{z}}}{\partial\xi_\ell}\;(\lambda\,,\,\xi)\right|^2} \;\;\ge\; [\,\bar C_7\;\cdot\bar\varepsilon_9\,]\cdot\frac{1}{\lambda_M}\;.$$

In (2.50) *and* (2.51), *the numbers* $\bar\varepsilon_8$, $\bar\varepsilon_9$, $\bar\varepsilon_{10}$ *and* $\bar C_7$ *are independent on* t *and* Δ *as long as* (2.47) *and* (2.48) *are fulfilled.*

Proof. As the situation is rotationally symmetric, we may assume that

$$\xi \;=\; (\,\delta\,,\,0\,,\,\cdots,\,0)\,, \qquad \text{where} \;\; \delta \,>\, 0\,.$$

Moreover,

$$(2.52) \;\; \lambda \;=\; \lambda_M \,+\, s \quad \text{and} \quad \lambda_M^{-1}\,|\,s| \le \bar\varepsilon_{10} \quad \Longrightarrow \quad (1 \,-\, \bar\varepsilon_{10})\,\lambda_M \;\le\; \lambda \;\le\; (1 \,+\, \bar\varepsilon_{10})\,\lambda_M\,,$$

In formula (2.38), we let

$$\varrho_b \quad \text{when} \quad R \;=\; R_b = t \,+\, \Delta\,; \qquad \text{and} \quad \text{by} \quad \varrho_c \quad \text{when} \quad R \;=\; R_c \;=\; t \,-\, \Delta\,.$$

Clearly ϱ_b and ϱ_c depend on λ and ξ. Using (2.43) and a argument similar to that in (2.45), we obtain

$$(2.53) \qquad (2\,\varrho)^2 \;=\; \frac{(4R_{\lambda_M})^2}{(1+R^2_{\lambda_M})^2} \;+\; O\left(\frac{s}{\lambda_M}\right) \;+\; O\,(\delta_\lambda^2)$$

$$\Longrightarrow \quad \varrho_b \;=\; \frac{\lambda_M}{t} + O\left(\frac{s}{\lambda_M}\right) \;+\; O\,(\delta_\lambda^2) \quad \text{and} \quad \varrho_c \;=\; \frac{\lambda_M}{t} \;+\; O\left(\frac{s}{\lambda_M}\right) \;+\; O\,(\delta_\lambda^2)$$

$$\text{for} \quad \delta_\lambda \quad \text{and} \quad \lambda_M^{-1}\,|\,s| \quad \text{small}\,, \quad \text{and} \;\; R_{\lambda_M} \le \sqrt{\frac{5}{2}}$$

$$\Longrightarrow \quad \varrho_b \;=\; \varrho_c \;+\; O\left(\frac{s}{\lambda_M}\right) \;+\; O\,(\delta_\lambda^2\,)\,.$$

Applying (2.48) and (2.49), we have

$$1-\frac{\Delta}{t} \;\ge\; a^2 \quad \Longrightarrow \quad \frac{R_b}{\lambda} \;=\; \frac{R_b}{\lambda_M\,+\,s} \;=\; \frac{R_b}{\lambda_M} \;+\; O\left(\frac{|\,s|}{\lambda_M}\right) \qquad [\,\text{cf.} \;\; (2.52)\,]$$

$$= \sqrt{\frac{t+\Delta}{t-\Delta}} \;+\; O\left(\frac{|\,s|}{\lambda_M}\right) \;=\; \sqrt{1+\frac{2\,(\Delta/t)}{1-(\Delta/t)}} \;+\; O\left(\frac{|\,s|}{\lambda_M}\right)$$

$$\Longrightarrow \quad \frac{R_b}{\lambda} \;\ge\; \sqrt{1+\frac{2\,b^2}{a^2}} \;+\; O\left(\frac{|\,s|}{\lambda_M}\right)\,.$$

Observe that $\dfrac{R_b}{\lambda_M} = \sqrt{\dfrac{t\,+\,\Delta}{t\,-\,\Delta}} \le \sqrt{\dfrac{5}{2}}$. Thus if we choose $\bar\varepsilon_8$ and $\bar\varepsilon_{10}$ small enough (they depend on $\mathbf{a}$ and b), we can find a positive number c such that

$$\frac{R_b}{\lambda} \;\ge\; 1\,+\,c^2 \quad \Longrightarrow \quad B_o(1) \;\subset\; B_{\frac{-\xi}{\lambda}}\,(\,\lambda^{-1}R_b\,)\,.$$

Hence the boundary sphere $\partial\,\dot{\mathcal{P}}^{-1}\left(B_{\frac{-\xi}{\lambda}}\,(\,\lambda^{-1}R_b)\right)$ lies inside the northern hemisphere. Let us denote its radius (measured in the Euclidean metric in $\mathbb{R}^{n+1}$) by ϱ_{b_λ}. Together with the upper bound in (2.48), we can find positive constants c_1 and c_2 so that

$$(2.54) \qquad 1\,-\,c_1^2 \;\ge\; r_{b_\lambda} \;\ge\; c_2^2 \;>\; 0\,.$$

Likewise, the boundary sphere

$$(2.55) \qquad \partial\,\dot{\mathcal{P}}^{-1}\left(B_{\frac{-\xi}{\lambda}}\,(\,\lambda^{-1}R_c)\right)$$

lies inside the southern hemisphere when we suitably choose $\bar\varepsilon_8$ and $\bar\varepsilon_{10}$. Denote by ϱ_{c_λ} the radius (again measured in the Euclidean metric in $\mathbb{R}^{n+1}$) of the sphere in (2.55). Similarly, via (2.47) and (2.48), there are positive numbers c_3 and c_4 such that

$$(2.56) \qquad 1\,-\,c_3^2 \;\ge\; r_{b_\lambda} \;\ge\; c_4^2 \;>\; 0\,.$$

Thus we are justified to use (2.10), which, together with (2.45) and (2.46), yield

$$\frac{\partial G_{|\mathbf{z}}}{\partial \xi_1}(\lambda\,,\,\xi) \;=\; \frac{\|\,S^{n-1}\|}{2}\left\{ \frac{\varrho_{b_\lambda}^{n-2}}{\sqrt{1-\varrho_{b_\lambda}^2}}\cdot\left[-\left.\frac{d\,[\,\varrho\,(\delta)]^2}{d\,\delta}\right|_{\varrho\,=\,\varrho_{b_\lambda}}\right] \right.$$

$$\left. +\;\frac{\varrho_{c_\lambda}^{n-2}}{\sqrt{1-\varrho_{c_\lambda}^2}}\cdot\left[-\left.\frac{d\,[\,\varrho\,(\delta)]^2}{d\,\delta}\right|_{\varrho\,=\,\varrho_{c_\lambda}}\right]\right\}$$

$$= -\,\frac{\|\,S^{n-1}\|}{2}\frac{\varrho_{b_\lambda}^{n-2}}{\sqrt{1-\varrho_{b_\lambda}^2}}\left\{ \left.\frac{d\,[\,\varrho(\delta)]^2}{d\,\delta}\right|_{\varrho\,=\,\varrho_{b_\lambda}} + \left.\frac{d\,[\,\varrho(\delta)]^2}{d\,\delta}\right|_{\varrho\,=\,\varrho_{c_\lambda}}\right\} + O\left(\frac{\delta_\lambda}{\lambda}\right)\left[O\left(\frac{s}{\lambda_M}\right)+O\,(\delta_\lambda^2)\right]$$

$$= \frac{4^4\|\,S^{n-1}\|}{2}\cdot\frac{\delta_\lambda}{\lambda}\cdot\frac{\varrho_{b_\lambda}^{n-2}}{\sqrt{1-\varrho_{b_\lambda}^2}}\cdot\frac{\Delta^2\,(t+\Delta)(t-\Delta)}{4\,t^4} \;+\; O\left(\frac{\delta_\lambda}{\lambda}\right)\left[O\left(\frac{s}{\lambda_M}\right)+O\,(\delta_\lambda^2)\right].$$

It follows from (2.47), (2.48), (2.54) and (2.56) that

$$\frac{r_{b_\lambda}^{n-2}}{\sqrt{1-r_{b_\lambda}^2}}\cdot\frac{\Delta^2\,(t\,+\,\Delta)(t-\Delta)}{4\,t^4} \;\;\ge\;\; \frac{c_2^{2(n-2)}}{\sqrt{1-(1-c_1)^2}}\cdot\frac{b^4}{4}\cdot\Big[\,1-(1-a^2)^2\Big] \;\ge\; c_5^2 \;>\; 0$$

Hence we have

$$\left|\frac{\partial\,G_{|\mathbf{z}}}{\partial\,\xi_1}(\lambda\,,\,\xi)\right| \;\ge\; c_6^2\cdot\delta_\lambda\cdot\frac{1}{\lambda_M} \qquad \text{for}\quad \lambda_M^{-1}\,\delta \quad\text{and}\quad \lambda_M^{-1}\,|\,s| \quad\text{small}\qquad \left(\delta_\lambda=\frac{\delta}{\lambda}\right).$$

Combining with (2.34) and (2.50) we obtain (2.51). □

§ **2 h.** *Derivative in* λ. Let us express (2.38) in terms of λ :

$$(2.57)\qquad (2\,\varrho)^2 \;=\; \frac{(4R_\lambda)^2\,[\,(1\,+\,R_\lambda^2\,-\,\delta_\lambda^2)^2\,+\,\delta_\lambda^2]}{[(R_\lambda\,+\,\delta_\lambda)^2\,+\,1\,]^2\,[(R_\lambda\,-\,\delta_\lambda)^2\,+\,1\,]^2} \qquad \left(R_\lambda\;=\;\frac{R}{\lambda}\,,\quad \delta_\lambda\;=\;\frac{\delta}{\lambda}\right)$$

$$= \frac{\lambda^2\,(4R)^2\,[\,(\lambda^2+R^2-\delta^2)^2+\delta^2\,\lambda^2\,]}{[(R+\delta)^2+\lambda^2]^2\,[(R-\delta)^2+\lambda^2]^2} \;=\; \frac{\lambda^2\,(4R)^2\,[\,(\lambda^2+R^2\,-\,\delta^2)^2+\delta^2\,\lambda^2\,]}{[\,(R^2\,+\,\delta^2\,+\,\lambda^2)^2\,-\,4R^2\delta^2\,]^2}\,.$$

Cf. the second last line in (2.41). The expression suggests that $\varrho\to 0$ as $\lambda\to 0^+$ (or $\lambda\to\infty$), corresponding to pushing the boundary sphere toward the north (resp. south) pole. (2.57) guides us to

$$(2.58)\qquad \frac{\partial\,(2\,r)^2}{\partial\lambda} \;=\; \frac{2\lambda\,(4R)^2}{\{[(R+\delta)^2+\lambda^2]\cdot[(R-\delta)^2+\lambda^2]\,\}^2}\;\times$$

$$\times\left[(\lambda^2+R^2-\delta^2)^2+2\lambda^2(\lambda^2+R^2)\;-\;\frac{[\,4\,\lambda^2][\,(\lambda^2+R^2-\delta^2)^2+\delta^2\,\lambda^2]\cdot[R^2+\delta^2+\lambda^2]}{[(R+\delta)^2+\lambda^2]\cdot[(R-\delta)^2+\lambda^2]}\right].$$

We note that when $\delta\;=\;0$ (i.e., $\xi\;=\;0$) and

$$\lambda\;=\;\lambda_M \qquad (\,\text{recall that}\;\;\lambda_M\;=\;\sqrt{(t\,+\,\Delta)(t\,-\,\Delta)}\,,\quad t\;>\;\Delta)\,,$$

(2.56) provides the following information. When $R = R_b := t + \Delta$:

$$\frac{\partial\,[(2\,\varrho)^2]}{\partial\lambda}\Big|_{\lambda = \lambda_M} = \frac{(2\lambda_M)\cdot(4R_b)^2}{(R_b^2+\lambda_M^2)^4} \times \Big[\,(R_b^2+\lambda_M^2)^2 + 2\lambda_M^2(R_b^2+\lambda_M^2) - 4\lambda_M^2(R_b^2+\lambda_M^2)\Big]$$

$$= \frac{(2\lambda_M)\cdot(4R_b)^2}{(R_b^2+\lambda_M^2)^3}\,(R_b^2-\lambda_M^2) = \frac{8\,\Delta\,\lambda_M}{t^3}\,.$$

When $R = R_c := t - \Delta$:

$$\frac{\partial\,[(2\,r)^2]}{\partial\lambda}\Big|_{\lambda=\lambda_M} = \frac{(2\lambda_M)\cdot(4R_c)^2}{(R_c^2+\lambda_M^2)^3}\,(R_c^2-\lambda_M^2) = -\,\frac{8\,\Delta\,\lambda_M}{t^3}\,. \tag{2.59}$$

Cf. (2.40). Combining with the information on the radius of the caps obtained in (2.39), together with (2.10), the changes at the boundaries of the two caps cancel each other. This provides another way to see (2.20) and Lemma 2.21.

As the situation is rotationally symmetric, we continue to assume that we arrange $\xi = (\delta\,,\ 0\,,\cdots,\ 0)$, where $\delta \ge 0$. With the domain Ω as in (2.1) being an annular domain, clearly, $G_{|\mathbf{z}}(\lambda,\ \xi) = G_{|\mathbf{z}}(\lambda,\ -\xi)$. It follows that

$$G_{|\mathbf{z}}(\lambda,\ \xi) = G_{|\mathbf{z}}(\lambda,\ 0) + O\,(|\,\xi|^2) \qquad \text{when} \quad |\,\xi| = |\,\delta| \quad \text{is} \quad \text{small}\,.$$

In what follows, we provide a more formal argument.

Expansion for small δ. Rewrite (2.58) in terms of R_λ $(= R\,\lambda^{-1})$ and δ_λ $(= \delta\,\lambda^{-1})$:

$$\frac{\partial\,(2\,r)^2}{\partial\lambda} = \frac{1}{\lambda}\cdot\frac{2\,(4R_\lambda)^2}{\{[(R_\lambda+\delta_\lambda)^2\,+\,1\,]\cdot[(R_\lambda-\delta_\lambda)^2+1\,]\,\}^2} \times \tag{2.60}$$

$$\times\left[(1+R_\lambda^2-\delta_\lambda^2)^2 \,+\, 2\,(1+R_\lambda^2) \,-\, \frac{4\,[\,(1+R_\lambda^2-\delta_\lambda^2\,)^2\,+\,\delta_\lambda^2\,]\times[R_\lambda^2+\delta_\lambda^2\,+\,1\,]}{[\,(R_\lambda+\delta_\lambda)^2\,+\,1\,]\cdot[(R_\lambda-\delta_\lambda)^2\,+\,1\,]}\right].$$

Using the expansion in (2.43) we derive the following [cf. also (2.40)].

$$\frac{\partial\,(2\,r)^2}{\partial\lambda} = \frac{2}{\lambda}\cdot\frac{(4R_\lambda)^2}{(1+R_\lambda^2)^3}\cdot[\,(R_\lambda^2-1)+O\,(\delta_\lambda^2)\,] \tag{2.61}$$

$$\left(= \frac{(2\lambda)\cdot(4R)^2}{(\lambda^2+R^2)^3}\cdot\Big[\,(R^2-\lambda^2)+(\lambda^2)\cdot O\,(\delta_\lambda^2)\,\Big]\right) \quad \text{for}\ \ \delta_\lambda \ge 0\ \ \text{small}\,,\ \ \text{here}\ R_\lambda \le \sqrt{\frac{5}{2}} + c^2.$$

In the above formula, c is a fixed and small positive number. Comparing with (2.40) and using the discussion in § 2 f, we recognize that the leading order term in (2.61) is the derivative in the unperturbed case. Combining with (2.10), we find that (for λ close to λ_M)

$$\frac{\partial G_{|\mathbf{z}}}{\partial\lambda}\,(\lambda\,,\ \xi) = \frac{\partial G_{|\mathbf{z}}}{\partial\lambda}\,(\lambda\,,\ 0) + \frac{1}{\lambda}\,O\,(\delta_\lambda^2) \quad \text{for}\ \ \delta_\lambda \ge 0\ \ \text{small}\,,\quad R_\lambda \le \sqrt{\frac{5}{2}}+c^2\,. \tag{2.62}$$

Applying Lemma 2.26 to (2.62), we obtain the following result.

Lemma 2.63. *Under the notations and the conditions in Lemma* 2.21, *let* $\delta = |\xi|$ *and* $\lambda = \lambda_M + s$, *and assume* (2.48), *there exist positive constants* $\bar{\varepsilon}_{11}$, $\bar{\varepsilon}_{12}$, $\bar{\varepsilon}_{13}$, *and* $\bar{C}_8$, *such that if*

$$\lambda_M^{-1} |\delta| \leq \bar{\varepsilon}_{11}, \qquad \bar{\varepsilon}_{12} \geq \lambda_M^{-1} |s| \geq \bar{\varepsilon}_{13} > 0, \tag{2.64}$$

then we have

$$\left| \frac{\partial G_{|\mathbf{z}}}{\partial \lambda} (\lambda_M + s, \xi) \right| \geq \bar{C}_8 \cdot \bar{\varepsilon}_{13} \cdot \frac{1}{\lambda_M}.$$

Here the positive constant $\bar{C}_8$ *is independent on* t, Δ *and* λ *as long as* (2.48) *and* (2.64) *are fulfilled.*

§ 2 i. *Stability under perturbation.* Recall that we denote the critical point by

$$\mathbf{p_M} = (\lambda_M, \vec{0}), \qquad \text{where} \quad (\text{as usual}) \quad \lambda_M = \sqrt{(t + \Delta)(t - \Delta)}.$$

Lemma 2.65. *Under the notations and the conditions in Lemma* 2.21, *assume also conditions* (2.47) *and* (2.48). *We can find positive numbers* $\gamma \in (0, 1)$ *and* $\bar{C}_9$ *such that*

$$\min_{\partial B_{\mathbf{p_M}}(\gamma \lambda_M)} \| \nabla G_{|\mathbf{z}}(\lambda, \xi; \Omega) \| \geq \bar{C}_9 \cdot \gamma \cdot \frac{1}{\lambda_M}, \qquad \textit{where}$$

$$\partial B_{\mathbf{p_M}}(\gamma \lambda_M) = \{ (\lambda, \xi) \in \mathbb{R}^+ \times \mathbb{R}^n \mid |s|^2 + |\xi|^2 = (\gamma \cdot \lambda_M)^2, \ \textit{with} \ s = \lambda - \lambda_M \}.$$

In addition, the constants γ *and* $\bar{C}_9$ *do not depend on* t *and* Δ *as long as* (2.47) *and* (2.48) *are fulfilled.*

Proof. We take γ to be small so that

$$(\lambda, \xi) \in \partial B_{\mathbf{p_M}}(\gamma \lambda_M) \implies \sqrt{\frac{5}{2}} + c^2 \geq \frac{t + \Delta}{\lambda} > \frac{t - \Delta}{\lambda} \geq \sqrt{\frac{2}{5}} - c^2 > 0.$$

For a point $(\lambda, \xi) \in \partial B_{\mathbf{p_M}}(\gamma \lambda_M)$, with $s = \lambda - \lambda_M$ and $|\delta| = |\xi|$, we have

$$\text{either} \quad |s| > \frac{\gamma}{\sqrt{2}} \lambda_M \quad \text{and} \quad |\delta| \leq \frac{\gamma}{\sqrt{2}} \lambda_M \implies \left| \frac{\partial G_{|\mathbf{z}}(\lambda, \xi; \Omega)}{\partial \lambda} \right| \geq \frac{C \gamma}{\lambda_M},$$

where we use Lemma 2.63;

$$\text{or} \qquad |\delta| > \frac{\gamma}{\sqrt{2}} \lambda_M \quad \text{and} \quad |s| \leq \frac{\gamma}{\sqrt{2}} \lambda_M \implies \| \nabla_\xi G_{|\mathbf{z}}(\lambda, \xi; \Omega) \| \geq \frac{C' \gamma}{\lambda_M}$$

(via Lemma 2.49). Thus we have the desired estimate. □

§3. Infinite number of solutions.

We begin by understanding the effect caused by other annular domain $B_o(R) \setminus \overline{B_o(\rho)}$ at a point $(\lambda_c\,,\ \vec{0}\,)$ (can be thought of as a critical point for another annular domain). As before, $R > \rho$ are given positive numbers.

§3 a. *C^o– effect: the case $\lambda_c^{-1} R \gg 1$ and $\lambda_c^{-1}\rho \gg 1$.* Let $\dfrac{R}{\lambda_c} > \dfrac{\rho}{\lambda_c} \gg 1$. In this case the annular domain, when pulled back to S^n via $\dot{\mathcal{P}}$, shrinks around the *north* pole. To be more precise, we consider the following computation.

$$\text{Let}\quad \tan\theta_R = \frac{R}{\lambda_c} \iff \theta_R := \arctan\left(\frac{R}{\lambda_c}\right)$$

$$\Longrightarrow \sin\left(\frac{\pi}{2} - \theta_R\right) = \left(1 + \frac{R^2}{\lambda_c{}^2}\right)^{-\frac{1}{2}} = O\left(\frac{\lambda_c}{R}\right)$$

$$\Longrightarrow \sin\left(\frac{\pi}{2} - \theta_R\right) = \frac{\lambda_c}{R}\left(1 + \frac{\lambda_c}{R}\right)^{-\frac{1}{2}} = \frac{\lambda_c}{R}\left[1 + O\left(\frac{\lambda_c}{R}\right)\right],$$

$$(3.1)\quad \varphi_R := \frac{\pi}{2} - \theta_R \Longrightarrow \sin\varphi_R = \varphi_R\,[\,1 + O\,(\varphi_R)] = \varphi_R\left[1 + O\left(\frac{\lambda_c}{R}\right)\right]$$

$$\Longrightarrow \varphi_R = \frac{\lambda_c}{R}\left[1 + O\left(\frac{\lambda_c}{R}\right)\right] \qquad \text{for} \quad \frac{R}{\lambda_c} \quad \text{large}\,.$$

It follows that

$$(3.2)\quad 0 < \sin 2\,\theta_R = \sin(\pi - 2\theta_R) = \sin 2\left(\frac{\pi}{2} - \theta_R\right) \le 2\sin\left(\frac{\pi}{2} - \theta_R\right) = O\left(\frac{\lambda_c}{R}\right)\,.$$

Likewise, we define θ_ρ is a similar fashion, and obtain

$$(3.3)\quad 0 < \sin 2\theta_\rho = \sin(\pi - 2\theta_\rho) = \sin 2\left(\frac{\pi}{2} - \theta_\rho\right) \le 2\sin\left(\frac{\pi}{2} - \theta_\rho\right) = O\left(\frac{\lambda_c}{\rho}\right)$$

for $\lambda_c^{-1}\rho \gg 1$. It follows that

$$(3.4)\qquad \int_{B_o(R)\setminus B_o(\rho)} \left(\frac{\lambda_c}{\lambda_c{}^2 + |\,y|^2}\right)^n$$

$$= \frac{\|S^n\|}{2^n}\int_{2\arctan\frac{\rho}{\lambda_c}}^{2\arctan\frac{R}{\lambda_c}} [\sin\theta'\,]^{n-1}\,d\theta' \qquad (\text{here } \ |\,y| = \tan\theta\,, \ \ \theta' = 2\,\theta\,)$$

$$\le C\left(\frac{\lambda_c}{\rho}\right)^{n-1}\cdot\left[\arctan\left(\frac{R}{\lambda_c}\right) - \arctan\left(\frac{\rho}{\lambda_c}\right)\right] \qquad [\,\text{via (3.2) \& (3.3)}\,]$$

$$\le C\left(\frac{\lambda_c}{r}\right)^{n-1}\cdot\left\{\left[\frac{\pi}{2} - \arctan\left(\frac{\rho}{\lambda_c}\right)\right] - \left[\frac{\pi}{2} - \arctan\left(\frac{R}{\lambda_c}\right)\right]\right\}$$

$$\leq \; C_1 \left(\frac{\lambda_c}{\rho}\right)^{n-1} \cdot \left[\left(\frac{\lambda_c}{\rho}\right) - \left(\frac{\lambda_c}{R}\right) + O\left(\frac{{\lambda_c}^2}{\rho^2}\right) \right] \qquad [\,\text{using} \;\; (3.2) \;\&\; (3.3)\,]$$

$$\leq \; C_2 \left(\frac{\lambda_c}{\rho}\right)^{n-1} \cdot \frac{\lambda_c\,(R-\rho)}{R\,\rho} \qquad \text{when} \;\; \lambda_c^{-1} R \; > \; \lambda_c^{-1}\rho \; \gg \; 1\,.$$

§3 b. *C^o–effect: the case $\lambda_c^{-1} R \;\approx\; 0$ and $\lambda_c^{-1}\rho \;\approx\; 0$.* Let

$$(3.5) \qquad \varepsilon^2 \;\geq\; \frac{R}{\lambda_c} \;>\; \frac{\rho}{\lambda_c} \;>\; 0\,, \qquad \text{where } \varepsilon \text{ is a (small) number}\,.$$

The situation is upside down (in S^n) of the consideration in §3 a.

$$(3.6) \quad \int_{B_o(R)\setminus B_o(\rho)} \left(\frac{\lambda_c}{{\lambda_c}^2 + |\,y\,|^2}\right)^n \;\leq\; C\,\frac{\|S^n\|}{2^n} \cdot \int_{2\arctan\frac{\rho}{\lambda_c}}^{2\arctan\frac{R}{\lambda_c}} [\,\theta'\,]^{\,n-1}\, d\theta' \qquad (\,\theta' = 2\,\theta)$$

$$\leq \; C' \left[\left(\frac{R}{\lambda_c}\right)^n - \left(\frac{\rho}{\lambda_c}\right)^n \right] \qquad \text{(domain gathers around the south pole)}$$

$$\leq \; C'' \left(\frac{R}{\lambda_c}\right)^{n-1} \left[\left(\frac{R}{\lambda_c}\right) - \left(\frac{\rho}{\lambda_c}\right) \right] = \frac{C''\cdot R^{n-1}\,(R-\rho)}{\lambda_c^n} \qquad \text{for} \;\; \varepsilon^2 \;\geq\; \frac{R}{\lambda_c} \;>\; \frac{\rho}{\lambda_c} \;>\; 0\,.$$

§3 c. *C^1–effect.* Based on (6.6) in Part I [16], we have

$$\left|\,\frac{\partial G_{|\mathbf{z}}}{\partial \xi_j}\,(\lambda_c\,,\; \vec{0})\right| \;\leq\; C_1 \int_{B_o(R)\setminus B_o(\rho)} \frac{{\lambda_c}^n\cdot|\,y\,|}{({\lambda_c}^2 + |\,y\,|^2)^{n+1}} \;=\; C_2 \int_\rho^R \frac{{\lambda_c}^n \cdot r^n\, dr}{({\lambda_c}^2 + r^2)^{n+1}}$$

$$= \; \frac{C_2}{\lambda_c} \cdot \int_{\arctan\frac{r}{\lambda_c}}^{\arctan\frac{R}{\lambda_c}} [\,\cos\,\theta]^n\,[\,\sin\,\theta]^n\, d\theta = \frac{C_3}{\lambda_c} \cdot \int_{2\arctan\frac{r}{\lambda_c}}^{2\arctan\frac{R}{\lambda_c}} [\,\sin\,\theta'\,]^n\, d\theta'$$

for $j = 1\,,\; 2\,,\; \cdots,\; n$. In the above, $r = \tan\theta$ and $\theta' = 2\,\theta$. We can carry out similar argument found in § 3 a and § 3 b. As in (3.4), we have

$$(3.7) \quad \int_{B_o(R)\setminus B_o(\rho)} \frac{{\lambda_c}^n\cdot r}{(\lambda_c^2 + |\,y\,|^2)^{n+1}} \;\leq\; \frac{\bar{C}}{\lambda_c}\cdot\left(\frac{\lambda_c}{\rho}\right)^n \frac{\lambda_c\,(R-\rho)}{R\,\rho} \quad \text{for} \;\; \lambda_c^{-1} R \gg 1\,,\;\; \lambda_c^{-1}\rho \gg 1\,.$$

On the other hand,

$$(3.8) \quad \int_{B_o(R)\setminus B_o(\rho)} \frac{{\lambda_c}^n\cdot r}{({\lambda_c}^2 + |\,y\,|^2)^{n+1}} \;\leq\; \frac{C''}{\lambda_c}\left(\frac{R}{\lambda_c}\right)^n \left[\left(\frac{R}{\lambda_c}\right) - \left(\frac{\rho}{\lambda_c}\right) \right]$$

$$= \; \frac{C''}{\lambda_c} \cdot \frac{R^n\,(R-\rho)}{{\lambda_c}^{n+1}} \qquad \text{for} \;\; \lambda_c^{-1} R \;\text{ and }\; \lambda_c^{-1}\rho \;\text{ small}\,.$$

Likewise, using (6.1) in Part I [16], we provide similar estimates for $\left|\,\dfrac{\partial G_{|\mathbf{z}}}{\partial \lambda}\,(\lambda_c\,,\; \vec{0})\right|$. Observe that the first term inside the bracket in (6.1) is similar to the integral in (3.4)

and (3.6). Knowing that this becomes the dominating term, we have the following results.

$$(3.9)\qquad \left\| \bigtriangledown \left[\int_{B_o(R)\setminus B_o(\rho)} \left(\frac{\lambda}{\lambda^2 + |\, y - \xi|^2} \right)^n \right]_{(\lambda_c\,,\ \vec{0})} \right\| \ \le\ \frac{C}{\lambda_c} \left(\frac{\lambda_c}{\rho} \right)^{n-1} \frac{1}{R} \cdot \frac{R-\rho}{\rho} \cdot \lambda_c$$

$$\text{for}\ \ \lambda_c^{-1} R \gg 1 \ \ \text{and}\ \ \lambda_c^{-1} \rho \gg 1\,.$$

$$(3.10)\qquad \left\| \bigtriangledown \left[\int_{B_o(R)\setminus B_o(\rho)} \left(\frac{\lambda}{\lambda^2 + |\, y - \xi|^2} \right)^n \right]_{(\lambda_c\,,\ \vec{0})} \right\| \ \le\ \frac{C}{\lambda_c} \cdot \left(\frac{R}{\lambda_c} \right)^{n-1} \cdot (R - \rho) \cdot \frac{1}{\lambda_c}$$

$$\text{for}\ \ \lambda_c^{-1} R \ \ \text{and}\ \ \lambda_c^{-1} \rho \ \ \text{small}\,.$$

§ 3 d. *Effect from* ξ. Consider the expression:

$$\left(\frac{\lambda_c}{\lambda_c^2 + |\, y - \xi|^{\,2}} \right)^n = \left[\frac{\lambda_c}{\lambda_c^2 + |\,y|^{\,2} - 2\, y \cdot \xi + |\,\xi|^{\,2}} \right]^n = \left(\frac{\lambda_c}{\lambda_c^2 + |\,y|^{\,2}} \right)^n \cdot \left[\frac{1}{1 + \frac{|\,\xi|^{\,2} - 2\, y\cdot \xi}{|\,y|^{\,2} + \lambda_c^2}} \right]^n .$$

Case one: there exist positive numbers c_1 and C such that

$$\lambda_c^{-1} |\,\xi| \le c_1 \quad \text{and} \quad \lambda_c^{-1} \rho \ge C \ \Longrightarrow\ \left| \frac{|\,\xi|^{\,2} - 2\, y \cdot \xi}{|\,y|^{\,2} + \lambda_c^2} \right| \ \le\ \frac{1}{2}$$

$$\Longrightarrow\quad \left(\frac{\lambda_c}{\lambda_c^2 + |\, y - \xi|^{\,2}} \right)^n \ \le\ 2^n \cdot \left(\frac{\lambda_c}{\lambda_c^2 + |\,y|^{\,2}} \right)^n \qquad \text{for}\ \ \rho \le |\,y| \le R.$$

Case two: there exist positive numbers c_1 and ε such that

$$\lambda_c^{-1} |\,\xi| \le c_1 \ \ \text{and}\ \ \lambda_c^{-1} R \le \varepsilon \ \Longrightarrow \left| \frac{|\,\xi|^{\,2} - 2\, y \cdot \xi}{|\,y|^{\,2} + \lambda_c^2} \right| = \left| \frac{\lambda_c^{-2} |\,\xi|^2 - 2\, (\lambda_c^{-1} y) \cdot (\lambda_c^{-1} \xi)}{\lambda_c^{-2} |\,y|^2 + 1} \right| \ \le\ \frac{1}{2}$$

$$\Longrightarrow\quad \left(\frac{\lambda_c}{\lambda_c^2 + |\, y - \xi|^{\,2}} \right)^n \ \le\ 2^n \cdot \left(\frac{\lambda_c}{\lambda_c^2 + |\,y|^{\,2}} \right)^n \qquad \text{for}\ \ \rho \le |\,y| \le R.$$

Under the condition $\lambda_c^{-1} |\,\xi| \le c_1$, the argument leading to (3.9) and (3.10) can be applied to the situation $\xi \ne 0$ after multiplying the estimates by a fixed constant. Moreover, so far we assume that $|\,H| \le 1$. The results can be rescaled correspondingly to $|\,H| \le \mathcal{A}$ for any positive constant $\mathcal{A}$.

Lemma 3.11. *Given any positive numbers* R *and* ρ *with* $R > \rho$, *in* (1.3), *assume* $|\,H| \le \mathcal{A}$ *in the annular domain* $B_o(R) \setminus \overline{B_o(\rho)}$, *and that* H *is equal to zero outside. Then we can find positive constants* c, ε , *and* C *such that for*

$$\text{(3.12)} \qquad \lambda_c^{-1}\,|\,\xi| \,\le\, c\,, \quad \lambda_c^{-1}\,R \,\ge\, C \quad \textit{and} \quad \lambda_c^{-1}\,\rho \,\ge\, C$$

$$\Longrightarrow \left\| \bigtriangledown \left(\int_{B_o(R)\,\setminus B_o(\rho)} H(y) \left(\frac{\lambda}{\lambda^2 \,+\, |\,y \,-\, \xi|^2} \right)^n \right) \Big|_{(\lambda_c\,,\ \xi)} \right\| \,\le\, \frac{C_1}{\lambda_c} \left(\frac{\lambda_c}{\rho} \right)^{n-1} \frac{\mathcal{A}}{R} \cdot \frac{R \,-\, \rho}{\rho} \cdot \lambda_c\,.$$

$$\text{(3.13)} \qquad \lambda_c^{-1}\,|\,\xi| \,\le\, c\,, \quad \varepsilon \,\ge\, \lambda_c^{-1}\,R \,>\, 0 \quad \textit{and} \quad \varepsilon \,\ge\, \lambda_c^{-1}\,\rho \,>\, 0$$

$$\Longrightarrow \left\| \bigtriangledown \left(\int_{B_o(R)\,\setminus B_o(\rho)} H(y) \left(\frac{\lambda}{\lambda^2 \,+\, |\,y \,-\, \xi|^2} \right)^n \right) \Big|_{(\lambda_c\,,\ \xi)} \right\| \le \frac{C_2}{\lambda_c} \cdot \left(\frac{R}{\lambda_c} \right)^{n-1} \cdot (R - \rho) \cdot \mathcal{A} \cdot \frac{1}{\lambda_c}\,.$$

Moreover, the positive constants C_1 *and* C_2 *do not depend on* $\lambda_c\,,\ \xi\,,\ R$ *and* ρ *as long as the conditions in* (3.12) *and* (3.13) *are fulfilled* (*respectively*).

§ 3 e. *Effect from changing* H. Suppose that H and H_1 are bounded and L^2-integrable on $\mathbb{R}^n$, and

$$\text{(3.14)} \qquad \int_{\mathbb{R}^n} |\,H(y) - H_1(y)|^2 \ \ \le\ \ \varepsilon^n \cdot \left(\sup_{\mathbb{R}^n} |\,H| \right)^2 .$$

To distinguish the reduced functionals, we denote by

$$\text{(3.15)} \qquad G_{|\mathbf{z}}(H_1)\,(\mathbf{z}) \ = \ \bar c_{-1} \int_{\mathbb{R}^n} H_1(y) \left[\frac{\lambda}{\lambda^2 + |\,y - \xi\,|^2} \right]^n \qquad \text{for } \ \mathbf{z} \,=\, V_{\lambda,\,\xi} \,\in\, \mathbf{Z}\,.$$

Likewise, we define $G_{|\mathbf{z}}(H)$.

C^o*– estimate.* Applying the Hölder inequality, we have

$$\begin{aligned}
\text{(3.16)} \qquad & |\,G_{|\mathbf{z}}(H) - G_{|\mathbf{z}}(H_1)| \\
= \ & \left|\ \bar c_{-1} \int_{\mathbb{R}^n} H(y) \left(\frac{\lambda}{\lambda^2 + |\,y - \xi\,|^2} \right)^n \ - \ \bar c_{-1} \int_{\mathbb{R}^n} H_1(y) \left(\frac{\lambda}{\lambda^2 + |\,y - \xi\,|^2} \right)^n \ \right| \\
\le \ & |\,\bar c_{-1}| \cdot \left[\int_{\mathbb{R}^n} |\,H(y) - H_1(y)|^2 \right]^{\frac{1}{2}} \cdot \left[\int_{\mathbb{R}^n} \left(\frac{\lambda}{\lambda^2 + |\,y - \xi\,|^2} \right)^{2n} \right]^{\frac{1}{2}} \\
\le \ & \bar C_{o,\ n} \cdot \left(\frac{\varepsilon}{\lambda} \right)^{\frac{n}{2}} \cdot \left(\sup_{\mathbb{R}^n} |\,H| \right) .
\end{aligned}$$

Precisely, $\bar C_{o,\ n} \ = \ |\,\bar c_{-1}| \cdot \left(\|\,S^{n-1}\| \int_0^{\frac{\pi}{2}} [\sin\theta]^{\,n-1}\ [\cos\theta]^{\,3\,(n-1)}\ d\,\theta \right)^{\frac{1}{2}}.$

C^1*– estimate.* Consider the first derivative in ξ. Refer to (6.3) in Part I [16]. As in (3.16), we have

$$(3.17) \quad \left| \frac{\partial G_{|\mathbf{z}}(H)}{\partial \xi_j} - \frac{\partial G_{|\mathbf{z}}(H_1)}{\partial \xi_j} \right|$$
$$\leq |\bar{c}_{-1}| \cdot \left[\int_{\mathbb{R}^n} |H(y) - H_1(y)|^2 \right]^{\frac{1}{2}} \left[\int_{\mathbb{R}^n} \frac{\lambda^{2n} |\xi_j - y_j|^2}{(\lambda^2 + |y - \xi|^2)^{2n+2}} \right]^{\frac{1}{2}}$$
$$\leq C \varepsilon^{\frac{n}{2}} \cdot \left(\sup_{\mathbb{R}^n} |H| \right) \left[\int_{\mathbb{R}^n} \frac{\lambda^{2n} |\xi - y|^2}{(\lambda^2 + |y - \xi|^2)^{2n+2}} \right]^{\frac{1}{2}}$$
$$= C' \varepsilon^{\frac{n}{2}} \cdot \left(\sup_{\mathbb{R}^n} |H| \right) \left[\int_0^\infty \frac{\lambda^{2n} r^{n+1}\, dr}{(\lambda^2 + r^2)^{2n+2}} \right]^{\frac{1}{2}}$$
$$= C' \varepsilon^{\frac{n}{2}} \cdot \left(\sup_{\mathbb{R}^n} |H| \right) \left[\int_0^{\frac{\pi}{2}} \frac{\lambda^{3n+2} [\tan \theta]^{n+1} \sec^2 \theta \, d\theta}{\lambda^{4(n+1)} [\sec^2 \theta]^{2n+2}} \right]^{\frac{1}{2}} = \bar{C}_{1,\, n} \left(\frac{\varepsilon}{\lambda} \right)^{\frac{n}{2}} \frac{1}{\lambda} \cdot \sup_{\mathbb{R}^n} |H|.$$

Here $j = 1, \cdots, n$, and $\bar{C}_{1,\, n}$ has a similar expression as $\bar{C}_{o,\, n}$ [in (3.16)]. From here, we know that by counting the power of λ and r, the other expressions in the first derivative of λ can be estimated accordingly, and it is also of order $O\left(\left[\frac{\varepsilon}{\lambda} \right]^{\frac{n}{2}} \cdot \frac{1}{\lambda} \right)$. We summarize the discussion of this section in the following.

Lemma 3.18. *Suppose that H and H_1 are bounded and L^2 functions on $\mathbb{R}^n$. Assume that* (3.14) *holds. Then*

$$(3.19) \qquad \left\| \bigtriangledown \left[G_{|\mathbf{z}}(H) - G_{|\mathbf{z}}(H_1) \right] \Big|_{(\lambda,\, \xi)} \right\| \leq C \left(\frac{\varepsilon}{\lambda} \right)^{\frac{n}{2}} \cdot \left(\sup_{\mathbb{R}^n} |H| \right) \cdot \frac{1}{\lambda}$$

for all $(\lambda, \xi) \in \mathbb{R}^+ \times \mathbb{R}^n$.

§ 3 f. *Spacing.* For a number $\mathbf{a} > 1$, let the annular domains be set up at

$$(3.20) \qquad B_o \left(\frac{1 + \eta}{\mathbf{a}} \right) \Big\backslash \overline{B_o \left(\frac{1 - \eta}{\mathbf{a}} \right)}, \quad \cdots, \quad B_o \left(\frac{1 + \eta}{\mathbf{a}^k} \right) \Big\backslash \overline{B_o \left(\frac{1 - \eta}{\mathbf{a}^k} \right)}, \quad \cdots.$$

Given positive numbers τ and η (to be specified later), define

$$(3.21) \quad H(y) = \frac{1}{\mathbf{a}^{\tau k}} \qquad \text{for} \quad y \in B_o \left(\frac{1 + \eta}{\mathbf{a}^k} \right) \Big\backslash \overline{B_o \left(\frac{1 - \eta}{\mathbf{a}^k} \right)}, \qquad k = 1, 2, \cdots;$$

$$H(y) = 0 \qquad \text{for} \quad y \notin \bigcup_{k=1}^{\infty} \left\{ B_o \left(\frac{1 + \eta}{\mathbf{a}^k} \right) \setminus \overline{B_o \left(\frac{1 - \eta}{\mathbf{a}^k} \right)} \right\}.$$

We first observe that if we choose $\eta \in (0, 1)$ so that

$$(3.22) \quad (1 + \eta) < (1 - \eta)\, \mathbf{a} \quad \Longrightarrow \quad \frac{1 + \eta}{\mathbf{a}^{k+1}} < \frac{1 - \eta}{\mathbf{a}^k} \quad \text{and} \quad \frac{1 + \eta}{\mathbf{a}^k} < \frac{1 - \eta}{\mathbf{a}^{k-1}}.$$

(That is, the next annular domain is surrounded by the present one.)

For each individual annular domain indexed by a positive integer m, the critical point appears in $(\lambda_{M_m}, \ \vec{0})$, where

$$
(3.23) \qquad \lambda_{M_m} := \sqrt{\frac{1+\eta}{\mathbf{a}^m}\cdot\frac{1-\eta}{\mathbf{a}^m}} = \frac{\sqrt{1-\eta^2}}{\mathbf{a}^m} \qquad (\text{via Lemma 2.21}).
$$

For the reader's convenience, we describe the effects of other annular domains first, before smoothing out the "edges".

Effects from outside annular domains. Here we fix $m \geq 2$. We prepare to apply (3.12) in Lemma 3.11 to estimate the C^1-effect. For a point $(\lambda, \ \xi) \in \mathbb{R}^+ \times \mathbb{R}^n$ with

$$
|\lambda - \lambda_{M_m}| \leq c\,\lambda_{M_m} \quad [\text{i.e. } (1-c)\,\lambda_{M_m} \leq \lambda \leq (1+c)\,\lambda_{M_m}], \quad \text{and} \quad \lambda_{M_m}^{-1}|\xi| \leq c,
$$

where $c \in (0, \ 1)$ is a constant, we obtain

$$
\begin{aligned}
&\lambda^{-1} \times (\text{inner radius of the nearest outside annular domain})\\
\geq\ & \frac{1}{1+c}\cdot\frac{1-\eta}{\mathbf{a}^{m-1}}\cdot\frac{1}{\lambda_{M_m}} = \frac{\mathbf{a}}{1-c}\cdot\sqrt{\frac{1-\eta}{1+\eta}}\,.
\end{aligned}
$$

When $\mathbf{a}$ is large enough, c is small enough, and $\eta < 1/2$, we can apply (3.12) in Lemma 3.11 to obtain

$$
\begin{aligned}
(3.24) \qquad & \sum_{k=1}^{m-1} \left\| \nabla \left[\int_{B_o\left(\frac{1+\eta}{\mathbf{a}^k}\right) \setminus \overline{B_o\left(\frac{1-\eta}{\mathbf{a}^k}\right)}} H(y)\cdot\left(\frac{\lambda}{\lambda^2+|y-\xi|^2}\right)^n \right] \Bigg|_{(\lambda, \ \xi)} \right\| \\
\leq\ & \frac{C}{\lambda}\sum_{k=1}^{m-1}\left\{\left[\frac{\lambda}{(1-\eta)\,\mathbf{a}^{-k}}\right]^{n-1}\cdot\left[\frac{\lambda}{(1+\eta)\,\mathbf{a}^{-k}}\right]\left(\frac{2\eta}{1-\eta}\right)\cdot\left[\frac{1}{\mathbf{a}^{\tau k}}\right]\right\}\\
\leq\ & \frac{C_{\eta, c}}{\lambda_{M_m}}\sum_{k=1}^{m-1}\left\{\left(\frac{1}{\mathbf{a}^m}\right)^n\cdot(\mathbf{a}^k)^n\cdot\left[\frac{1}{\mathbf{a}^{\tau k}}\right]\right\}\\
=\ & \frac{C_{\eta, c}}{\lambda_{M_m}}\cdot\frac{1}{\mathbf{a}^{mn}}\sum_{k=1}^{m-1}(\mathbf{a}^k)^{n-\tau} = \frac{C_{\eta, c}}{\lambda_{M_m}}\cdot\frac{1}{\mathbf{a}^{mn}}\sum_{k=1}^{m-1}[\mathbf{a}^{n-\tau}]^k\\
=\ & \frac{C_{\eta, c}}{\lambda_{M_m}}\cdot\frac{1}{\mathbf{a}^{mn}}\cdot\frac{[\mathbf{a}^{n-\tau}]^m-\mathbf{a}^{n-\tau}}{\mathbf{a}^{n-\tau}-1} \leq \frac{C_{\eta, c}}{\lambda_{M_m}}\cdot\frac{1}{\mathbf{a}^{mn}}\cdot[\mathbf{a}^{n-\tau}]^{m-1} \quad (\text{provided } \mathbf{a}^{n-\tau} > 2)\\
=\ & \left(\frac{1}{\lambda_{M_m}}\cdot\frac{1}{\mathbf{a}^{\tau m}}\right)\cdot\frac{C_{\eta, c}}{\mathbf{a}^{n-\tau}} \quad \text{for} \quad 0 < \eta < \frac{1}{2}, \ |\lambda - \lambda_{M_m}| \leq c\,\lambda_{M_m} \ \& \ \lambda_{M_m}^{-1}|\xi| \leq c\,.
\end{aligned}
$$

Roughly speaking, the first term in the last expression above is the strength of the gradient from the contribution of the m-th annular domain (cf. Lemma 2.65). Once we take $\tau > n$, the last term in the above, that is, $C_{\eta, \ c}/\mathbf{a}^{n-\tau}$, can be made small when we choose $\mathbf{a}$ large enough (maintaining $0 < \eta < 1/2$).

Effects from inner annular domains. Similarly,

$$\begin{aligned} &\lambda^{-1} \cdot (\text{outer radius of the nearest inside annular domain}) \\ \le\; & \frac{1}{1 - c} \cdot \frac{1 + \eta}{\mathbf{a}^{m+1}} \cdot \frac{1}{\lambda_{M_m}} \;=\; \frac{1}{1 - c} \cdot \frac{1}{\mathbf{a}} \cdot \sqrt{\frac{1 + \eta}{1 - \eta}}\,. \end{aligned}$$

When $\mathbf{a}$ is large enough, c is small enough, and $\eta < 1/2$, we apply (3.13) in Lemma 3.11 to form the following estimate.

$$(3.25) \qquad \sum_{k=m+1}^{\infty} \left\| \bigtriangledown \left(\int_{B_o\left(\frac{1+\eta}{\mathbf{a}^k}\right) \setminus \overline{B_o\left(\frac{1-\eta}{\mathbf{a}^k}\right)}} H(y) \cdot \left(\frac{\lambda}{\lambda^2 + |\, y - \xi|^2} \right)^n \right) \Big|_{(\lambda,\, \xi)} \right\|$$

$$\begin{aligned} &\le \frac{C'}{\lambda} \sum_{k=m+1}^{\infty} \left\{ \left[\frac{(1+\eta)\,\mathbf{a}^{-k}}{\lambda} \right]^{n-1} \cdot \left(\frac{2\,\eta\,\mathbf{a}^{-k}}{\lambda} \right) \cdot \left[\frac{1}{\mathbf{a}^{\tau k}} \right] \right\} \;\le\; \frac{C'_{\eta,\,c}}{\lambda_{M_m}} \cdot \mathbf{a}^{m\,n} \cdot \sum_{k=m+1}^{\infty} \left[\frac{1}{\mathbf{a}^{n+\tau}} \right]^k \\ &\le \frac{C'_{\eta,\,c}}{\lambda_{M_m}} \cdot \mathbf{a}^{m\,n} \left[\frac{1}{\mathbf{a}^{n+\tau}} \right]^{m+1} \cdot \frac{1}{1 - \frac{1}{\mathbf{a}^{n+\tau}}} \;\le\; \left(\frac{1}{\lambda_{M_m}} \cdot \frac{1}{\mathbf{a}^{\tau m}} \right) \cdot \frac{C'_{\eta,\,c}}{\mathbf{a}^{n+\tau+1}} \end{aligned}$$

for $0 < \eta < \frac{1}{2}$, $|\,\lambda - \lambda_{M_m}| \le c\,\lambda_{M_m}$ and $\lambda_{M_m}^{-1}\,|\xi| \le c$. Likewise, the last term $C'_{\eta\,,\,c} \,/\, \mathbf{a}^{n+\tau+1}$ can be made small when we choose $\mathbf{a}$ large.

§ 3 g. *Smoothing out the edges.* We thicken the annular domain by bringing in the adjustment factor $\sigma > 0$:

$$(3.26) \qquad B_o \left(\frac{1 + (\eta + \sigma)}{\mathbf{a}^k} \right) \Big\backslash\, \overline{B_o \left(\frac{1 - (\eta + \sigma)}{\mathbf{a}^k} \right)}\,.$$

Here $1 - (\eta + \sigma) > 0$. Recall that H is defined in (3.21). We make use of the extra space available in (3.26) to smooth out the inner and outer 'edges'. To do that, fix a C^∞-function $\mathcal{F}$ on $\mathbb{R}^n$ with the following properties.

$$\begin{aligned} (3.27) \qquad \mathcal{F}(y) &= 1 \qquad \text{for } 1 - \eta \le |y| \le 1 + \eta\,, \\ \mathcal{F}(y) &= 0 \qquad \text{for } |y| \ge 1 + (\eta + \sigma) \;\text{ or }\; |y| \le 1 - (\eta + \sigma)\,, \\ 1 \ge \mathcal{F}(y) &\ge 0 \quad \text{for } 1 - (\eta + \sigma) < |y| < 1 - \eta \;\text{ or }\; 1 + (\eta + \sigma) \ge |y| \ge 1 + \eta\,. \end{aligned}$$

For an integer $m \ge 1$, let $\varsigma = \mathbf{a}^{-m}$. Then consider the rescaling

$$\begin{aligned} (3.28) \qquad & \mathcal{S}(y) = \frac{1}{\mathbf{a}^{\tau\, m}} \cdot \mathcal{F}(\varsigma^{-1}\, y) \\ \Longrightarrow \qquad & \sup_{\mathbb{R}^n} \|\bigtriangledown^{(h)} \mathcal{S}\| \le \frac{1}{\mathbf{a}^{\tau\, m}} \cdot \frac{C(\sigma)}{\varsigma^k} = \frac{C(\sigma)}{\mathbf{a}^{m\,(\tau - h)}} \\ \Longrightarrow \qquad & \sup_{\mathbb{R}^n} \|\bigtriangledown^{(h)} \mathcal{S}\| \to 0 \quad \text{for } h \le n - 1 \quad \text{as} \quad m \to \infty \quad (n - 1 < \tau < n)\,. \end{aligned}$$

Denote the smoothened function by H^S, which has the following property.

$$H^S(y) = \frac{1}{\mathbf{a}^{\tau k}} \qquad \text{for } y \in B_o\left(\frac{1+\eta}{\mathbf{a}^k}\right) \setminus \overline{B_o\left(\frac{1-\eta}{\mathbf{a}^k}\right)}, \qquad k = 1, 2, \cdots; \tag{3.29}$$

$$0 \le H^S(y) \le \frac{1}{\mathbf{a}^{\tau k}} \quad \text{for } y \in B_o\left(\frac{1+(\eta+\sigma)}{\mathbf{a}^k}\right) \setminus \overline{B_o\left(\frac{1-(\eta+\sigma)}{\mathbf{a}^k}\right)},$$

$$H^S(y) = 0 \quad \text{for } y \notin \bigcup_{k=1}^{\infty} \left\{ B_o\left(\frac{1+(\eta+\sigma)}{\mathbf{a}^k}\right) \setminus \overline{B_o\left(\frac{1-(\eta+\sigma)}{\mathbf{a}^k}\right)} \right\}.$$

$$\| \nabla^{(h)} H^S(y) \| \to 0 \qquad \text{for} \quad h \le n-1 \quad \text{as} \quad |y| \to 0^+. \tag{3.30}$$

§ 3 h. *Regularity of* H^S *at* 0. It follows from (3.30) that H^S has $(n-1)$-th order of flatness at 0. As

$$n > \tau > (n-1) \implies \tau - (n-1) > 0, \tag{3.31}$$

we can gain a bit of Hölder regularity at the origin, arriving to $H^S \in C^{n-1,\,\beta}(\mathbb{R}^n)$. Currently, this appears to be the best balance between strength [cf. (3.29)] and spacing [cf. § 3 f and (2.48)].

§ 3 i. *Existence of an infinite number of stable critical points and the proof of Main Theorem* 1.7. Let us now fix $\tau \in (n-1, n)$, and make it precise that

$$\Omega_m := B_o\left(\frac{1+\eta}{\mathbf{a}^m}\right) \setminus \overline{B_o\left(\frac{1-\eta}{\mathbf{a}^m}\right)}, \tag{3.32}$$

$$H_m(y) := \frac{1}{\mathbf{a}^{\tau m}} \quad \text{for } y \in B_o\left(\frac{1+\eta}{\mathbf{a}^m}\right) \setminus \overline{B_o\left(\frac{1-\eta}{\mathbf{a}^m}\right)},$$

$$H_m(y) := 0 \quad \text{for } y \notin B_o\left(\frac{1+\eta}{\mathbf{a}^m}\right) \setminus \overline{B_o\left(\frac{1-\eta}{\mathbf{a}^m}\right)}.$$

Here $m \in \mathbb{N} \setminus \{0\}$. With definition (2.1), and via Lemma 2.21 and Lemma 2.23, $G_{|\mathbf{z}}(\bullet, \bullet; \Omega_m)$ has a non-degenerate critical point at

$$\mathbf{p}_{\mathbf{M_m}} := (\lambda_{M_m}, \vec{0}), \qquad \text{where} \qquad \lambda_{M_m} = \frac{\sqrt{(1+\eta)(1-\eta)}}{\mathbf{a}^m}. \tag{3.33}$$

It follows that

$$\frac{(1+\eta)\,\mathbf{a}^{-m}}{\lambda_{M_m}} = \sqrt{\frac{1+\eta}{1-\eta}}, \qquad \frac{(1-\eta)\,\mathbf{a}^{-m}}{\lambda_{M_m}} = \sqrt{\frac{1-\eta}{1+\eta}}, \qquad \frac{\eta \cdot \mathbf{a}^{-m}}{\mathbf{a}^{-m}} = \eta. \tag{3.34}$$

Cf. $$\frac{t+\Delta}{\lambda_M}, \qquad \frac{t-\Delta}{\lambda_M}, \qquad \frac{\Delta}{t} \qquad \text{in Lemma 2.65}.$$

We now find and fix $\eta \in (0,\ 1)$ so that

$$(3.35) \qquad 1 - A^2 \ > \ \eta \ > \ B^2 \ > \ 0 \quad \text{and} \quad \frac{5}{2} \ \geq \ \frac{1+\eta}{1-\eta} \ > \ \frac{1-\eta}{1+\eta} \ \geq \ \frac{2}{5}\,.$$

One can verify that with $(3.34)-(3.36)$, conditions (2.47) and (2.48) are fulfilled. (We emphasize that the choice of η is once and for all $m = 1,\ 2, \cdots.$)

After a rescaling ($H \ = \ 1 \ \to \ H \ = \ \mathbf{a}^{-\tau\, m}$), the conclusion in Lemma 2.65 implies that

$$(3.36) \qquad \min_{\partial B_{\mathbf{p}_{\mathbf{M_m}}}(\gamma\, \lambda_{M_m})} \| \bigtriangledown\ G_{|\mathbf{z}}(\lambda\,,\ \xi\,;\ \Omega_m)\,(\lambda\,,\ \xi)\| \ \geq \ \frac{\bar C_9\ \gamma}{\lambda_{M_m}} \cdot \frac{1}{\mathbf{a}^{\tau\, m}}\ .$$

Here the positive constants $\bar C_9$ and γ do not depend on m.

To keep the notation neat, we introduce $G_{|\mathbf{z}}(H_m)$ and $G_{|\mathbf{z}}(H^S)$ as in (3.15). *We claim that*

(3.37) "$G_{|\mathbf{z}}(H^S)$ *has a stable critical point inside* $B_{\mathbf{p}_{\mathbf{M_m}}}(\gamma\, \lambda_{M_m})$."

Let us begin with

$$(3.38) \qquad \left\| \bigtriangledown\ \{G_{|\mathbf{z}}(H^S) - G_{|\mathbf{z}}(H_m)\}\Big|_{(\lambda,\ \xi)} \right\|$$
$$\leq C_1 \left\{ \left\| \bigtriangledown \left(\int_{B_o\left(\frac{1+(\eta+\sigma)}{\mathbf{a}^m}\right) \setminus \overline{B_o\left(\frac{1-(\eta+\sigma)}{\mathbf{a}^m}\right)}} |\, H^S(y) - H\,(y)| \cdot \left(\frac{\lambda}{\lambda^2 + |\, y - \xi|^2} \right)^n \right) \Big|_{(\lambda,\ \xi)} \right\| \right.$$
$$+ \sum_{k=1}^{m-1} \left\| \bigtriangledown \left(\int_{B_o\left(\frac{1+(\eta+\sigma)}{\mathbf{a}^k}\right) \setminus \overline{B_o\left(\frac{1-(\eta+\sigma)}{\mathbf{a}^k}\right)}} H^S(y) \cdot \left(\frac{\lambda}{\lambda^2 + |\, y - \xi|^2} \right)^n \right) \Big|_{(\lambda,\ \xi)} \right\|$$
$$\left. + \sum_{k=m+1}^{\infty} \left\| \bigtriangledown \left(\int_{B_o\left(\frac{1+(\eta+\sigma)}{\mathbf{a}^k}\right) \setminus \overline{B_o\left(\frac{1-(\eta+\sigma)}{\mathbf{a}^k}\right)}} H^S(y) \cdot \left(\frac{\lambda}{\lambda^2 + |\, y - \xi|^2} \right)^n \right) \Big|_{(\lambda,\ \xi)} \right\| \right\}.$$

Starting with the first group in the right hand side of (3.38), we proceed with

$$\int_{B_o\left(\frac{1+(\eta+\sigma)}{\mathbf{a}^m}\right) \setminus \overline{B_o\left(\frac{1-(\eta+\sigma)}{\mathbf{a}^m}\right)}} |\, H^S\,(y) \ - \ H_m\,(y)|^2$$
$$\leq \ \left(\frac{2}{\mathbf{a}^{\tau m}} \right)^2 \cdot \int_{B_o\left(\frac{1+(\eta+\sigma)}{\mathbf{a}^m}\right) \setminus \overline{B_o\left(\frac{1-(\eta+\sigma)}{\mathbf{a}^k}\right)}} dy \qquad\qquad [\text{using } (3.29)\ \&\ (3.30)]$$
$$\leq \ 4\,\|\, S^{n-1}\| \cdot \left[\int_{\frac{1+\eta}{\mathbf{a}^m}}^{\frac{1+(\eta+\sigma)}{\mathbf{a}^m}} r^{n-1}\, dr \ + \ \int_{\frac{1-(\eta+\sigma)}{\mathbf{a}^m}}^{\frac{1-\eta}{\mathbf{a}^m}} r^{n-1}\, dr \right] \cdot \left(\frac{1}{\mathbf{a}^{\tau\, m}} \right)^2$$
$$\leq \ C_2 \left(\frac{1}{\mathbf{a}^m} \right)^n \cdot \{\, [\, 1 + (\eta + \sigma)\,]^n - [\, 1 + \eta]^n \ + \ [1 - \eta]^n - [\, 1 - (\eta + \sigma)\,]^n\,\} \cdot \left(\frac{1}{\mathbf{a}^{\tau\, m}} \right)^2$$

$$\leq \; C_3 \left(\frac{1}{\mathbf{a}^m}\right)^n \cdot \sigma \times \left(\frac{1}{\mathbf{a}^{\tau\, m}}\right)^2 \qquad\qquad (\text{ for } \;\; \sigma > 0 \quad \text{small})$$

$$\leq \; \left(\frac{C_4\ \sigma^{\frac{1}{n}}}{\mathbf{a}^m}\right)^n \cdot \left(\sup_{\mathbb{R}^n} |\, H_m|\right)^2 \;\; \leq \;\; \left(C_5\ \sigma^{\frac{1}{n}} \cdot \lambda_{M_m}\right)^n \cdot \left(\sup_{\mathbb{R}^n} |\, H_m|\right)^2 \qquad [\,\text{via} \;\; (3.33)]\,.$$

The positive constant C_5 does not depend on m. Cf. condition (3.14), which is required in Lemma 3.18. The argument leading to Lemma 3.18 shows that

$$(3.39) \quad \left\| \bigtriangledown \left(\int_{B_o\left(\frac{1+(\eta+\sigma)}{\mathbf{a}^m}\right) \setminus \overline{B_o\left(\frac{1-(\eta+\sigma)}{\mathbf{a}^m}\right)}} |\, H^S\,(y) - H\,(y)| \cdot \left(\frac{\lambda}{\lambda^2 + |\, y - \xi|^2}\right)^n \right) \Big|_{(\lambda\,,\ \xi)} \right\|$$

$$\leq \;\; C_6 \sqrt{\sigma} \cdot \frac{1}{\lambda_{M_m}} \cdot \left\{\frac{1}{\mathbf{a}^{\tau\, m}}\right\} \qquad\qquad [\,\lambda_{M_m} \;\; \text{is} \;\; \text{expressed} \;\; \text{in} \;\; (3.33)\,]$$

$$\text{for} \;\; (\lambda\,,\ \xi) \in \overline{B_{\mathbf{p_{cm}}}(\gamma\,\lambda_{M_m})} \qquad [\;\Longrightarrow \;\; (1 \,-\, \gamma)\,\lambda_{M_m} \;\leq\; \lambda \;\leq\; (1 \,+\, \gamma)\,\lambda_{M_m}\,].$$

The second and third groups in the right hand side of (3.38) can be estimated as in (3.24) and (3.25) by changing $(1+\eta) \to [\,1 + (\eta + \sigma)\,]$ and $(1-\eta) \to [\,1 - (\eta+\sigma)\,]$ [when $\sigma > 0$ is small, the arguments in (3.24) and (3.25) are not affected]. This leads to

(3.40)

$$\left\| \bigtriangledown \; \{G_{|\mathbf{z}}(H^S) \, - G_{|\mathbf{z}}(H_m)\}\Big|_{(\lambda\,,\ \xi)} \right\| \;\; \leq \;\; \left[\frac{1}{\lambda_{M_m}} \cdot \frac{1}{\mathbf{a}^{\tau\, m}}\right] \cdot \left(C_6 \sqrt{\sigma} \; + \; \frac{C_7}{\mathbf{a}^{n-\tau}} + \frac{C_8}{\mathbf{a}^{n+\tau+1}}\right).$$

The positive constants do not depend on m, $\mathbf{a}$, η and η, as long as the conditions in (3.35) are fulfilled, and η is small enough. Recall that $n - 1 < \tau < n$. Hence we can choose '$\mathbf{a}$' to be large enough, and σ to be small enough, so that (3.40) yields

$$(3.41) \qquad \left\| \bigtriangledown \; \{G_{|\mathbf{z}}(H^S) \, - G_{|\mathbf{z}}(H_m)\}\Big|_{(\lambda\,,\ \xi)} \right\| \;\; \leq \;\; \frac{1}{2} \cdot \frac{\bar{C}_9\,\gamma}{\lambda_{M_m}} \cdot \frac{1}{\mathbf{a}^{\tau\, m}}$$

for $(\lambda\,,\ \xi) \in \overline{B_{\mathbf{p_{cm}}}(\gamma\,\lambda_{M_m})}$. (3.36) and (3.41) imply that the gradient estimate

$$(3.42) \quad \left\| \bigtriangledown \; \{G_{|\mathbf{z}}(H^S)\Big|_{(\lambda\,,\ \xi)} \right\| \;\; \geq \;\; \frac{1}{2} \cdot \frac{\bar{C}_9\,\gamma}{\lambda_{M_m}} \cdot \frac{1}{\mathbf{a}^{\tau\, m}} \qquad \text{for} \quad (\lambda\,,\ \xi) \in \partial B_{\mathbf{p_{cm}}}(\gamma\,\lambda_{M_m})\,.$$

Together with (2.25), and C^o property of degree [refer to [9]; cf. also (A.3.5), (A.3.6), (A.3.7) and (A.3.8) in the e-Appendix], we conclude that

$$\text{Deg}\,(\bigtriangledown G_{|\mathbf{z}}(H^S)\,,\; B_{\mathbf{p_{M_m}}}(\gamma\,\lambda_{M_m})\,,\;\; \vec{0}\,) \;=\; \text{Deg}\,(\bigtriangledown G_{|\mathbf{z}}(H_m)\,,\; B_{\mathbf{p_{M_m}}}(\gamma\,\lambda_{M_m})\,,\;\; \vec{0}\,) = -\,1\,.$$

(For more information on the degree, see, for example, § A.3.3 in the e-Appendix.) Hence $G_{|\mathbf{z}}(H^S)$ has a stable critical point [denoted by $(\lambda_m\,,\ \xi_m)$] inside $B_{\mathbf{p_{M_m}}}(\gamma\,\lambda_{M_m})$. {See [9]; cf. also (A.3.4) and § A.3.4 in the e-Appendix.}

As the above argument works for all $m \in \mathbb{N}$, we apply (3.42) in above, together with Theorem 4.26 and Lemma 5.4 in Part I [16], to deduce the existence part in Main Theorem 1.7. Moreover, (1.10) in the Main Theorem is a consequence of Theorem 4.17 in Part I [16] and the fact that the stable critical point $(\lambda_m, \xi_m) \in B_{\mathbf{p_{M_m}}}(\gamma\,\lambda_{M_m}) \Longrightarrow \lambda_m \to 0^+$ and $\xi_m \to \vec{0}$ as $m \to \infty$. It also follows that $|\xi_m| \le C\,\lambda_m$ for $m \ge 1$.

§ 3 j . *Back to S^n.* When we transfer the solutions $\{v_m\}_{m=1}^{\infty}$ back to S^n via the stereographic projection as solutions $\{u_m\}_{m=1}^{\infty}$ of equation (1.1), we need to show that the north pole $\mathbf{N}$ is a removable singularity. This follows from the fact that the solutions $\{v_m\}_{m=1}^{\infty}$ appeared in Main Theorem 1.7, where we obtain in § 3 h, have bounded $\|\ \|_{\triangledown}$ - norms (recall (2.8) in Part I [16]) . In fact, using the Sobolev inequality [(2.9) in Part I [16]], the Kelvin transform as used in the proof of Theorem 3.16 in [13], together with a result of Brezis and Kato [6], we show that $\mathbf{N}$ is a removable singularity. See § A.14 in the e-Appendix. From the form of $\dot{\mathcal{P}}$ [(2.1) in Part I [16]] , it can be seen that the south pole $\mathbf{S}$ is the blow - up point for the sequence $\{u_m\}_{m=1}^{\infty}$.

e -Appendix can be found in

www.math.nus.edu.sg/ ˜ matlmc/e -Appendix.pdf

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